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Semilinear Systems of Weakly Coupled Damped Waves

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Contents

Introduction

We are interested in this thesis to study semilinear classical damped wave models with a particular class of time-dependent dissipation. The model we have in mind is

$$u_{tt} - \Delta u + b(t)u_t = g(u), \ \ u(0,x) = u_0(x), u_t(0,x) = u_1(x), \tag{0.1}$$

and a generalization to a weakly coupled system

$$\begin{aligned} &u_{tt} - \Delta u + b_1(t)u_t = g(v), \quad v_{tt} - \Delta v + b_2(t)v_t = f(u), \\ &u(0,x) = u_0(x), u_t(0,x) = u_1(x), v(0,x) = v_0(x), v_t(0,x) = v_1(x), \end{aligned} \tag{0.2}$$

where $t \in [0,\infty)$, $x \in \mathbb{R}^n$, $f(0) = g(0) = 0$, and

$$|g(v) - g(\tilde{v})| \lesssim |v - \tilde{v}|(|v| - |\tilde{v}|)^{p-1}, \quad |f(u) - f(\tilde{u})| \lesssim |u - \tilde{u}|(|u| - |\tilde{u}|)^{q-1}.$$

0.1. What does means waves?

A wave is a time evolution phenomenon that we generally model mathematically by using partial differential equations (PDEs) which have a dependent variable $u(x,t)$ (representing the wave value), an independent variable time t and one or more independent spatial variables $x \in \mathbb{R}^n$. The actual form that the wave takes is strongly dependent upon the system, the initial conditions, the boundary conditions for the solution, the domain and any system disturbances. Waves occur in most scientific and engineering disciplines, for example: fluid mechanics, optics, electromagnetism, solid mechanics, structural mechanics, quantum mechanics, etc. The waves for all these applications are described by solutions to either linear or nonlinear PDEs.
We do not focus here on methods of solution for each type of wave equation, but rather we concentrate on a small selection of relevant topics. However, first, it is legitimate to ask: what actually is a wave? This is not a straight forward question to answer.

Now, whilst most people have a general notion of what a wave is, based on their everyday experience, it is not easy to formulate a definition that will satisfy everyone engaged in or interested in this wide ranging subject. In fact, many technical works related to waves eschew a formal definition altogether and introduce the concept by a series of examples; for example, physics of waves [14] and hydrodynamics [34]. Nevertheless, it is useful to at least make an attempt and a selection of various definitions from normally authoritative sources is given below:

- "A time-varying quantity which is also a function of position" (Chambers Dictionary of Science and technology).

- "$\cdots$ a wave is any recognizable signal that is transferred from one part of the medium to another with a recognizable velocity of propagation" (Linear and nonlinear Waves [66]).

- "Speaking generally, we may say that it denotes a process in which a particular state is continually handed on without change, or with only gradual change, from one part of a medium to another" (Encyclopedia Britannica 1911).

We can give more examples who gives another definitions, then this variety confirm that "wave" is indeed not an easy concept to define.
A list is given below of physical wave types with examples of occurrence and references where more details may be found.

- Acoustic waves: Audible sound, medical applications of ultrasound, underwater sonar applications [14].

- Chemical waves: Concentration variations of chemical species propagating in a system [56].

- Electromagnetic waves: Electricity in various forms, radio waves, light waves in optic fibers [58].

- Gravitational waves: The transmission of variations in a gravitational field in the form of waves, as predicted by Einstein's theory of general relativity. Undisputed verification of their existence is still awaited [48].

- Seismic Waves: Resulting from earthquakes in the form of $P-$waves and $S-$waves, large explosions, high velocity impacts [14].

- Traffic flow waves: Small local changes in velocity occurring in high density situations can result in the propagation of waves and even shocks [35].

- Water waves: We give some examples.

  1. Capillary waves (Ripples).

  2. Rossby (or planetary) waves.

  3. Shallow water waves.

  4. Ship waves.

  5. Tsunami waves.

Among the most remarkable phenomenon and models studied in waves is that one of damped waves. In the following section we introduce the notion of waves and damped waves in a mathematical way.

0.2. Damped wave equations

0.2.1. Background

Let us show at the beginning some results which help to construct a chronological knowledge of our main problem.

Classical wave equation

The classical wave equation has the following form

$$u_{tt} - \Delta u = 0. \tag{0.3}$$

This equation describes the propagation of waves. It appears in numerous models as for the vibrating string or membrane, the propagation of sound, the longitudinal vibrations of an elastic rod or beam, surface water waves, the propagation of electric signals or for the description of electric or magnetic fields.
In 1747 d'Alembert studied in his papers [10, 11] and [12] wave models in 1D case of the form

$$u_{tt} - u_{xx} = 0, \quad u(0,x) = u_0(x), u_t(0,x) = u_1(x).$$

D'Alembert found the *so-called d'Alembert's representation of solution*

$$u(t,x) = \frac{1}{2}\big(u_0(x-t) + u_0(x+t)\big) + \frac{1}{2}\int_{x-t}^{x+t} u_1(s)ds.$$

From d'Alembert's representation we remark some properties of solution for example: No loss of regularity for solutions with respect to the Cauchy data, finite speed of propagation of perturbations, Huygens' principle, propagation of singularities, domain of dependence. This gives a first hint to qualitative properties of solutions of hyperbolic equations. For wave models with sources or sinks one can use Duhamel's principle to obtain a representation of solutions.
In 1883 G. Kirchhoff in [51] introduced a representation of solutions in 3D case for free wave model. One can prove the representation of solution in 2D case by using the *method of descent*. The representation of solution in general dimension (odd and even) introduced in [3, 15, 70] and [71].
The notion **energy of solution** is a very effective tool for the treatment of non-stationary models, for example, for the model (0.3) we can define the energy as follows:

$$\begin{aligned} E(u)(t) &:= \tfrac{1}{2}\textstyle\int_{\mathbb{R}^n} \left(|u_t(t,x)|^2 + |\nabla u(t,x)|^2\right) dx \\ &= \tfrac{1}{2}\|u_t(t,\cdot)\|^2_{L^2(\mathbb{R}^n)} + \tfrac{1}{2}\|\nabla u(t,\cdot)\|^2_{L^2(\mathbb{R}^n)}. \end{aligned} \tag{0.4}$$

From this definition one can prove the conservation of energy and uniqueness results. One of the important tools to study classical wave model (0.3) in general dimension is the Fourier transform which allows us to obtain some well-posedness results for different classes of solutions.
Then using Fourier transform we can prove for $u_0 \in H^s$ and $u_1 \in H^{s-1}$ that the Cauchy problem (0.3) has a unique energy solution

$$u \in \mathcal{C}\big([0,\infty), H^s(\mathbb{R}^n)\big) \cap \mathcal{C}^1\big([0,\infty), H^{s-1}(\mathbb{R}^n)\big).$$

Damped wave equations

Many papers are concerned with the classical homogeneous damped wave equation, i.e., the particular case of (0.1) with $b \equiv 1$ and $g \equiv 0$ given by

$$u_{tt} - \Delta u + u_t = 0, \;\; u(0,x) = u_0(x), \;\; u_t(0,x) = u_1(x). \tag{0.5}$$

One can prove that the energy (0.4) to the problem (0.5) is decaying when $t \longrightarrow \infty$. The exact behavior of decay is described in [38] by Matsumura as follows:

$$\begin{aligned} \big\|D_t^k D_x^\alpha u(t,\cdot)\big\|_{L^2(\mathbb{R}^n)} \;\; &\lesssim \;\; (1+t)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{q}\right)-k-\frac{|\alpha|}{2}} \\ &\qquad \times \big(\|u_0\|_{H^{k+|\alpha|}(\mathbb{R}^n)} + \|u_1\|_{H^{k+|\alpha|-1}(\mathbb{R}^n)} + \|(u_0,u_1)\|_{L^m(\mathbb{R}^n)}\big), \end{aligned}$$

where $m \in [1,2]$.
Now we turn to semilinear damped wave equations given by

$$u_{tt} - \Delta u + u_t = f(u), \;\; u(0,x) = u_0(x), \;\; u_t(0,x) = u_1(x). \tag{0.6}$$

In [39], the authors proved for a given initial data $(u_0,u_1) \in H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ with compact support, and $p \le p_{GN}(n) = \frac{n}{n-2}$ if $n \ge 3$ that the Cauchy problem (0.6) admits a unique local solution $u \in \mathcal{C}([0,T), H^1(\mathbb{R}^n)) \cap \mathcal{C}^1([0,T), L^2(\mathbb{R}^n))$ for some maximal existence time T. One of the first results on the global existence to (0.6) was given in [39] establishing global existence for small data by using the technique of potential well and modified potential well.
Assuming compactly supported data $(u_0,u_1) \in H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ to be sufficiently small, a global existence result for $p > p_{Fuj}(n) = 1 + \frac{2}{n}$ and $p \le p_{GN}(n)$ was proved in [63] by using the Matsumura estimates for solutions to linear classical damped wave models.
The condition of the compact support of the data was weakend in [31] by assuming small data in a suitable weighted Sobolev space with norm given by

$$I^2 := \int_{\mathbb{R}^n} e^{\frac{|x|^2}{2}} \left(|u_1(x)|^2 + |\nabla u_0(x)|^2 + |u_0(x)|^2\right) dx < \epsilon.$$

The corresponding Cauchy problem (0.6) has a uniquely determined global (in time) small data energy solution

$$u \in \mathcal{C}\big([0,T), H^1(\mathbb{R}^n)\big) \cap \mathcal{C}^1\big([0,T), L^2(\mathbb{R}^n)\big),$$

satisfying the decay estimates

$$\begin{aligned} \|u_t(t,\cdot)\|_{L^2(\mathbb{R}^n)} + \|\nabla u(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim I(1+t)^{-\left(\frac{1}{2}+\frac{n}{4}\right)}, \\ \|u(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim I(1+t)^{-\frac{n}{4}}. \end{aligned}$$

In [30], the authors showed that the smallness in weighted Sobolev spaces or compactly supported data can be avoided by some assumptions to get global (in time) existence of solutions for $n = 1, 2, 3$. In [26] the additional regularity $L^m(\mathbb{R}^n)$ appeared with $m \in [1, 2]$.

Diffusion phenomena

The diffusion phenomenon between heat and classical damped wave models is one of the reasons to study estimates for solutions of (0.6). The main result is a remarkable relation between solutions of the heat and damped wave equation. In other words, solutions to damped waves seem to behave more like solutions of the heat equation at large times, which allows us to use estimates of solutions to heat models to understand some properties to the solutions of (0.6). It is interesting that the critical exponent obtained for semilinear classical damped wave equations with power nonlinearity is exactly equal to the Fujita critical exponent for the semilinear heat equation $u_t - \Delta u = |u|^p$.

In the following section we explain how diffusion phenomena play an important role in the classification of time-dependent dissipation terms, i.e., under which conditions for the initial data and dissipation term we will derive estimates which imply the diffusion phenomenon for more general damped wave equations.

Linear damped wave equations with time-dependent dissipation

Let us now consider the linear damped wave equation with time-dependent dissipation term

$$u_{tt} - \Delta u + b(t)u_t = 0, \;\; u(0,x) = u_0(x), \;\; u_t(0,x) = u_1(x). \tag{0.7}$$

In his PhD thesis [67] the author introduced a classification of time-dependent dissipation terms in the following way:

- Scattering states to the free wave equation producing (solutions behave like solutions to the free wave equation),
- non-effective dissipation producing,
- effective dissipation producing (Matsumura-type decay estimates),
- over-damping producing.

Non-effectivity means that asymptotic properties of solutions are still described by solutions to the free wave equation. In other words, the dissipation term has a weak influence on the behavior of solutions. The case of strong influence of dissipation term means the effective case which will be treated in this thesis. We will explain in detail conditions which describe the effectiveness of the dissipation term. A typical example for some coefficient in an effective damping term is $b(t) = \frac{\mu}{(1+t)^r}$ for some $\mu > 0$ and $r \in (-1,1)$. When the dissipation becomes too strong we call this case over-damping, for example, if $r > 1$, thus, we lose the decay of the energy.
We introduce the space

$$\mathcal{A}_{m,s} := \big(H^s(\mathbb{R}^n) \cap L^m(\mathbb{R}^n)\big) \times \big(H^{\max\{s-1;0\}}(\mathbb{R}^n) \cap L^m(\mathbb{R}^n)\big) \tag{0.8}$$

with the norm

$$\|(u,v)\|_{\mathcal{A}_{m,s}} := \|u\|_{H^s(\mathbb{R}^n)} + \|u\|_{L^m(\mathbb{R}^n)} + \|v\|_{H^{s-1}(\mathbb{R}^n)} + \|v\|_{L^m(\mathbb{R}^n)}, \tag{0.9}$$

where $s \in \mathbb{R}_+$ and $m \in [1,2]$. If $m = 2$, then we do not have any additional regularity. We restrict ourselves to the effective case. In [67] and [69] Matsumura-type estimates for solutions to (0.7) with data from the energy space were given in the following way:

$$\begin{aligned}
\big\|u(t,\cdot)\big\|_{L^2(\mathbb{R}^n)} &\lesssim \big(1+B(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)}\big\|(u_0,u_1)\big\|_{\mathcal{A}_{m,1}}, && (0.10)\\
\big\|\nabla u(t,\cdot)\big\|_{L^2(\mathbb{R}^n)} &\lesssim \big(1+B(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{1}{2}}\big\|(u_0,u_1)\big\|_{\mathcal{A}_{m,1}}, && (0.11)\\
\big\|u_t(t,\cdot)\big\|_{L^2(\mathbb{R}^n)} &\lesssim b(t)^{-1}\big(1+B(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-1}\big\|(u_0,u_1)\big\|_{\mathcal{A}_{m,1}}. && (0.12)
\end{aligned}$$

Semilinear damped wave equations with time-dependent dissipation

In order to prove the global (in time) existence of small data solutions to a given semilinear Cauchy problem after using Duhamel's principle we need the estimates of solutions to the following family of parameter-dependent Cauchy problems:

$$u_{tt} - \Delta u + b(t)u_t = 0, \;\; u(\tau,x) = 0, \;\; u_t(\tau,x) = g(\tau,x). \tag{0.13}$$

In 2013, the authors derived in [4] the following estimates of solutions to the Cauchy problem (0.13):

$$\begin{aligned}
\big\|u(t,\cdot)\big\|_{L^2(\mathbb{R}^n)} &\lesssim b(\tau)^{-1}\big(1+B(t,\tau)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)}\|g(\tau,\cdot)\|_{L^m(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)},\\
\big\|\nabla u(t,\cdot)\big\|_{L^2(\mathbb{R}^n)} &\lesssim b(\tau)^{-1}\big(1+B(t,\tau)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{1}{2}}\|g(\tau,\cdot)\|_{L^m(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)},\\
\big\|u_t(t,\cdot)\big\|_{L^2(\mathbb{R}^n)} &\lesssim b(t)^{-1}b(\tau)^{-1}\big(1+B(t,\tau)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-1}\|g(\tau,\cdot)\|_{L^m(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)}.
\end{aligned}$$

In the same paper they proved for $n \leq 4$ the global (in time) existence of small data energy solutions belonging to

$$\mathcal{C}\big([0,\infty), H^1(\mathbb{R}^n)\big) \cap \mathcal{C}^1\big([0,\infty), L^2(\mathbb{R}^n)\big),$$

where $p > p_{Fuj}(n)$ and $p \leq p_{GN}(n)$ for $n \geq 3$. The initial data (u_0, u_1) are assumed to belong to

$$\mathcal{A}_{1,1} = \big(H^1(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)\big) \times \big(L^2(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)\big).$$

The optimality of this result is proved in [9] which means that we have in general a blow-up result for the local (in time) Sobolev solutions if $p \leq p_{Fuj}(n)$.

0.2.2. Main results

In Chapter 1 of the thesis our main goal is to get a generalization of previous results for semilinear damped wave equations in two directions. On the one hand we try to get some benefits from some additional regularity for the data to expand the dimension n for which we have global (in time) existence of small data solutions. On the other hand we use different regularity of data in order to obtain a larger admissible range of exponents p which allow to prove a global (in time) existence result of small data Sobolev solutions. The first chapter is devoted to study the Cauchy problem (0.1) for $g(u) = |u|^p$ in four cases of regularity of data, that is, for given low regular data, for data belonging to the energy space, for data from Sobolev spaces with suitable higher regularity and, finally, for large regular data.

- In the first section we will show that if the data has a low regularity

  $$(u_0, u_1) \in \mathcal{A}_{m,s} = \big(H^s(\mathbb{R}^n) \cap L^m(\mathbb{R}^n)\big) \times \big(L^2(\mathbb{R}^n) \cap L^m(\mathbb{R}^n)\big),$$

  where $s \in (0,1)$ and $m \in [1,2)$, then we prove a global (in time) existence result of Sobolev solutions provided that the exponent p belongs to some admissible interval and satisfies

  $$p > p_{Fuj,m}(n) = 1 + \frac{2m}{n}.$$

- In the second section the data are taken from energy space with additional regularity, that is,

  $$(u_0, u_1) \in \mathcal{A}_{m,1} = \big(H^1(\mathbb{R}^n) \cap L^m(\mathbb{R}^n)\big) \times \big(L^2(\mathbb{R}^n) \cap L^m(\mathbb{R}^n)\big).$$

  Then we extend the results of [4] by using an additional regularity for the data. The global (in time) existence of small data energy solutions can be proved for larger dimension n, where $p > p_{Fuj,m}(n)$ and belongs to some admissible interval.

- In the third section we treat the Cauchy problem (0.1) for

  $$(u_0, u_1) \in \mathcal{A}_{m,s} = \big(H^s(\mathbb{R}^n) \cap L^m(\mathbb{R}^n)\big) \times \big(H^{s-1}(\mathbb{R}^n) \cap L^m(\mathbb{R}^n)\big),$$

  where $s > 1$. In addition to $p_{Fuj,m}(n)$ and after using the fractional chain rule from Section A.3 in Appendix (the reader can find more details in [54]) another condition comes into play which is

  $$p > \lceil s \rceil \tag{0.14}$$

  for any space dimension n.

- A particular case which is treated separately in the last section is the case $s > \frac{n}{2}+1$. Using fractional powers from Section A.4 in Appendix the condition (0.14) can be relaxed to $p > s$.

The first chapter is completed by some concluding remarks.

0.3. Systems of weakly coupled semilinear damped wave equations

In Chapters 2 and 3 of the thesis we apply the results obtained for single equation to weakly coupled systems of semilinear damped wave equations. Let us now recall some results related to the system (0.2).

0.3.1. Background

First, we consider the case where $b_1(t) = b_2(t) = 1$, $g(v) = |v|^p$ and $f(u) = |u|^q$ in (0.2), namely,

$$\begin{aligned} &u_{tt} - \Delta u + u_t = |v|^p, \qquad v_{tt} - \Delta v + v_t = |u|^q, \\ &u(0,x) = u_0(x), v(0,x) = v_0(x), u_t(0,x) = u_1(x), v_t(0,x) = v_1(x). \end{aligned} \tag{0.15}$$

In 2007, Sun and Wang have shown in [60] the existence and nonexistence of energy solution to (0.15), for $n = 1, 3$ provided that the following condition is satisfied:

$$\frac{n}{2} > \frac{\max\{p; q\} + 1}{pq - 1}. \tag{0.16}$$

Narazaki generalized in [40] these existence results to $n = 1, 2, 3$, and improved the time decay estimates when $n = 3$. Condition (0.16) is still valid and necessary. Recently in 2014, Nishihara and Wakasugi determined in [45] the critical exponent for any space dimension n where the proof of global (in time) existence of energy solutions for supercritical nonlinearities is based on a weighted energy method provided that (0.16) holds.

0.3.2. Main results

In Chapter 2 we assume $b_1(t) = b_2(t) = b(t)$, $g(v) = |v|^p$ and $f(u) = |u|^q$ in (0.2). So, we have the model

$$\begin{aligned} &u_{tt} - \Delta u + b(t)u_t = |v|^p, \qquad v_{tt} - \Delta v + b(t)v_t = |u|^q, \\ &u(0,x) = u_0(x), v(0,x) = v_0(x), u_t(0,x) = u_1(x), v_t(0,x) = v_1(x). \end{aligned} \tag{0.17}$$

From results for a single equation, where the data are taken from energy space or are low regular we remark that the pivotal condition for the exponent p is defined by the modified Fujita exponent $p_{Fuj,m}(n)$. For this reason we compare in system (0.17) the exponents p and q with $p_{Fuj,m}(n)$ in these cases. In the case where only one exponent is above $p_{Fuj,m}(n)$ we shall prove a global (in time) existence result with a loss of decay and the additional condition

$$\frac{n}{2} > m\Big(\frac{\max\{p;q\}+1}{pq-1}\Big). \tag{0.18}$$

We will also show some benefits to prove global (in time) existence results after choosing the data with different additional regularities (hint of Prof. M. D'Abbicco), namely, we suppose $(u_0,u_1)\in\mathcal{A}_{m_1,s}$ and $(v_0,v_1)\in\mathcal{A}_{m_2,s}$. In the case of high regular data we assume different regularities $(u_0,u_1)\in\mathcal{A}_{m,s_1}$ and $(v_0,v_1)\in\mathcal{A}_{m,s_2}$, where the conditions for global (in time) existence are influenced strongly by these regularities.
In Chapter 3 we assume the particular case of the system (0.2), where the source terms are $f(u)=|u|^q$ and $g(v)=|v|^p$ and the dissipation terms $b_1(t)u_t$ and $b_2(t)v_t$ have the time-dependent coefficients $b_1(t):=\frac{1}{(1+t)^{r_1}}$ and $b_2(t):=\frac{1}{(1+t)^{r_2}}$, for $r_1,r_2\in(-1,1)$. So, the model we have in mind is

$$\begin{aligned}&u_{tt}-\Delta u+\tfrac{1}{(1+t)^{r_1}}u_t=|v|^p, \qquad v_{tt}-\Delta v+\tfrac{1}{(1+t)^{r_2}}v_t=|u|^q,\\ &u(0,x)=u_0(x), u_t(0,x)=u_1(x), v(0,x)=v_0(x), v_t(0,x)=v_1(x).\end{aligned}$$

We treat similar cases as in Chapter 2 but now we feel the interaction of different dissipation terms. We collect the results in several tables corresponding to each case of regularity. Finally, we conclude the third chapter by some generalizations of the dissipation terms appearing in the last weakly coupled system.

Chapter 4 is devoted to the Weakly coupled systems of semilinear damped waves with different scale- invariant time-dependent dissipation terms, the model which we mean is

$$\begin{aligned}&u_{tt}-\Delta u+\tfrac{\mu_1}{1+t}u_t=|v|^p, \qquad v_{tt}-\Delta v+\tfrac{\mu_2}{1+t}v_t=|u|^q,\\ &u(0,x)=u_0(x),\ u_t(0,x)=u_1(x),\ v(0,x)=v_0(x),\ v_t(0,x)=v_1(x),\end{aligned} \tag{0.19}$$

where $(t,x)\in[0,\infty)\times\mathbb{R}^n$ and $\mu_1,\mu_2>1$ are real constants. We conclude from Chapter 4 that for μ_1 and μ_2 are sufficiently large the solution to (0.19) behave like the solution to the system with effective dissipation terms. On the contrary, for small μ_1 and μ_2 we should consider the above system as a non-effective one.

In Chapter 5 we present some Blow up results for weakly coupled systems of Semilinear damped waves. After showing the basics of test function method which we will used later for the system (0.17), we prove that the solution cannot defined globally (in time) under suitable assumptions for the data and the following condition is satisfies

$$\frac{n}{2}\le\frac{\max\{p;q\}+1}{pq-1}. \tag{0.20}$$

The last result confirm the optimality of our previous existence results.

1. Semilinear wave models with effective damping term and power nonlinearity

In this chapter, we recall some results for linear wave equations with time-dependent dissipation introduced by J. Wirth in his paper [69] and his PhD thesis [67]. We present results for a family of linear parameter-dependent wave equations with time-dependent dissipation proved by M. D'Abbicco, S. Lucente and M. Reissig in [4]. These estimates are an important tool to prove the global existence of small data energy solutions to the Cauchy problem

$$u_{tt} - \Delta u + b(t)u_t = |u|^p, \;\; u(0,x) = u_0(x), \;\; u_t(0,x) = u_1(x) \tag{1.1}$$

under the requirement that the data (u_0, u_1) belong to

$$\mathcal{A}_{1,1} = (H^1(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)) \times (L^2(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)).$$

The main concern of the authors was to find the critical exponent which coincides in this case with the classical Fujita exponent $p_{Fuj}(n) = 1 + \frac{2}{n}$. The main goal of this chapter is to generalize this result to the case where the data (u_0, u_1) are taken from the space

$$\mathcal{A}_{m,s} := (H^s(\mathbb{R}^n) \cap L^m(\mathbb{R}^n)) \times (H^{\max\{s-1;0\}}(\mathbb{R}^n) \cap L^m(\mathbb{R}^n)),$$

where $m \in [1,2)$ and $s > 0$. We show how the critical exponent, it is a modified Fujita exponent, depends on m and the dimension n. The choice of s influences the notion of solution, too. On the one hand we have Sobolev solutions, only, on the other hand we have energy or even classical solutions. We distinguish between the cases of low regular data $s \in (0,1)$, data producing energy solutions in the case $s = 1$ and data having a suitable higher regularity $s > 1$. In the latter case we distinguish between data from Sobolev spaces with suitable regularity $s \in \left(1, \frac{n}{2}+1\right]$, and large regular data with $s > \frac{n}{2}+1$.

1.1. Introduction and tools from linear theory

In this section we summarize some tools of the linear wave theory which we will apply in the following sections.

For this reason we address to the Cauchy problem

$$u_{tt} - \Delta u + b(t)u_t = 0, \ \ u(0,x) = u_0(x), \ \ u_t(0,x) = u_1(x). \tag{1.2}$$

In [67] the following classification of time-dependent damping terms $b(t)u_t$ was introduced:

1. Scattering states to the free wave producing.

2. Non-effective dissipation producing.

3. Effective dissipation producing.

4. Over-damping producing.

Let us recall the properties of $b = b(t)$ to define an effective dissipation term $b(t)u_t$.

Hypothesis 1.1. *We assume the following assumptions to the time-dependent coefficient $b = b(t)$:*

1. $b(t) > 0$ *for any* $t \geq 0$*,*

2. $b(t)$ *is a monotonic function with* $tb(t) \to \infty$ *as* $t \to \infty$*,*

3. $((1+t)^2 b(t))^{-1} \in L^1(0,\infty)$*,*

4. $b \in \mathcal{C}^3([0,\infty))$ *and*

$$|b^{(k)}(t)| \lesssim \frac{b(t)}{(1+t)^k} \ \text{ for } \ k = 1,2,3, \tag{1.3}$$

5. $\frac{1}{b} \notin L^1(0,\infty)$*,*

6. *there exists a constant* $a \in [0,1)$ *such that*

$$tb'(t) \leq ab(t). \tag{1.4}$$

Example 1.1. *(See [4, 67] and [69]) The following functions $b = b(t)$ satisfy the conditions of Hypothesis 1.1:*

1. $b(t) = \frac{\mu}{(1+t)^r}$ *for some* $\mu > 0$ *and* $r \in (-1,1)$,

2. $b(t) = \frac{\mu}{(1+t)^r}(\log(c_{r,\gamma}+t))^\gamma$ *for some* $\mu > 0$ *and* $\gamma > 0$*,*

3. $b(t) = \frac{\mu}{(1+t)^r(\log(c_{r,\gamma}+t))^\gamma}$ *for some* $\mu > 0$ *and* $\gamma > 0$.

Here $c_{r,\gamma}$ is a sufficiently large positive constant.

Definition 1.1.1. *We denote by $B(t,0)$ the primitive of $\frac{1}{b(t)}$ which vanishes at $t=0$, that is,*

$$B(t,0)=\int_0^t \frac{1}{b(r)}dr.$$

If the conditions (1) and (5) of Hypothesis 1.1 are satisfied, then $B(t,0)$ is positive, strictly increasing, and $B(t,0)\to\infty$ as $t\to\infty$.

Definition 1.1.2. *We denote by $B(t,\tau)$ the primitive of $\frac{1}{b(t)}$ which vanishes at $t=\tau$, that is,*

$$B(t,\tau)=\int_\tau^t \frac{1}{b(r)}dr=B(t,0)-B(\tau,0).$$

Lemma 1.1. *Thanks to (1.3) and (1.4) the primitive $B(t,\tau)$ satisfies the following properties which will be used later:*

$$\begin{aligned} B(t,\tau) &\approx \frac{t}{b(t)}-\frac{\tau}{b(\tau)} \text{ for all } \tau\in[0,t], && (1.5)\\ B(t,\tau) &\approx B(t,0) \text{ for all } \tau\in\Big[0,\frac{t}{2}\Big], && (1.6)\\ B(\tau,0) &\approx B(t,0) \text{ for all } \tau\in\Big[\frac{t}{2},t\Big], && (1.7)\\ 1+t &\approx b(t)(1+B(t,0)), && (1.8)\end{aligned}$$

$$\int_{\frac{t}{2}}^t \frac{1}{b(\tau)}\big(1+B(t,\tau)\big)^{-\frac{j}{2}-l}d\tau \lesssim (1+B(t,0))^{1-\frac{j}{2}-l}\log\left(1+B(t,0)\right)^l \text{ for } j+l=0,1. \quad (1.9)$$

For the proof see [4].

Theorem 1.1. *(Data from energy space) ([67, 69])*
If the assumptions (1) to (5) of Hypothesis 1.1 are satisfied and if the data (u_0,u_1) belong to $\mathcal{A}_{m,1}$, then the energy solution to the Cauchy problem (1.2) satisfies the following estimates:

$$\begin{aligned} \big\|u(t,\cdot)\big\|_{L^2(\mathbb{R}^n)} &\lesssim \big(1+B(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)}\big\|(u_0,u_1)\big\|_{\mathcal{A}_{m,1}}, && (1.10)\\ \big\|\nabla u(t,\cdot)\big\|_{L^2(\mathbb{R}^n)} &\lesssim \big(1+B(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{1}{2}}\big\|(u_0,u_1)\big\|_{\mathcal{A}_{m,1}}, && (1.11)\\ \big\|u_t(t,\cdot)\big\|_{L^2(\mathbb{R}^n)} &\lesssim b(t)^{-1}\big(1+B(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-1}\big\|(u_0,u_1)\big\|_{\mathcal{A}_{m,1}}. && (1.12)\end{aligned}$$

Theorem 1.2. *(Low regular data)*
If the assumptions (1) to (5) of Hypothesis 1.1 are satisfied and if the data (u_0,u_1) belong to $\mathcal{A}_{m,s}$ for $s\in(0,1)$, then the Sobolev solution to the Cauchy problem (1.2) satisfies the following estimates:

$$\big\|u(t,\cdot)\big\|_{L^2(\mathbb{R}^n)} \lesssim \big(1+B(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)}\big\|(u_0,u_1)\big\|_{\mathcal{A}_{m,s}}, \quad (1.13)$$

$$\big\||D|^s u(t,\cdot)\big\|_{L^2(\mathbb{R}^n)} \lesssim \big(1+B(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s}{2}}\big\|(u_0,u_1)\big\|_{\mathcal{A}_{m,s}}. \quad (1.14)$$

Proof. To prove these statements it is sufficient to prove (1.14) because (1.13) was already proved in [67]. In order to use an interpolation argument we have from estimate (1.10) the following mapping L which is defined as follows:

$$L : (u_0, u_1) \in \mathcal{A}_{m,0} \longrightarrow u(t,\cdot) \in \dot{H}^0(\mathbb{R}^n)$$

with

$$\|L\|_{\mathcal{A}_{m,0}\longrightarrow \dot{H}^0(\mathbb{R}^n)} \lesssim \big(1+B(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)};$$

the estimate (1.11) implies

$$L : (u_0, u_1) \in \mathcal{A}_{m,1} \longrightarrow u(t,\cdot) \in \dot{H}^1(\mathbb{R}^n)$$

with

$$\|L\|_{\mathcal{A}_{m,1}\longrightarrow \dot{H}^1(\mathbb{R}^n)} \lesssim \big(1+B(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{1}{2}}.$$

From Proposition A.7 in the Appendix we conclude that the operator L also maps $\mathcal{A}_{m,s}$ into $\dot{H}^s(\mathbb{R}^n)$ for $s \in (0,1)$ with the norm estimate

$$\|L\|_{\mathcal{A}_{m,s}\longrightarrow \dot{H}^s(\mathbb{R}^n)} \lesssim \|L\|^{1-\theta}_{\mathcal{A}_{m,0}\longrightarrow L^2(\mathbb{R}^n)} \|L\|^{\theta}_{\mathcal{A}_{m,1}\longrightarrow \dot{H}^1(\mathbb{R}^n)}, \tag{1.15}$$

where $s = (1-\theta)\cdot 0 + \theta \cdot 1$. Then after straight-forward calculations we arrive at the estimate (1.14). $\square$

Theorem 1.3. *(High regular data)*
If the assumptions (1) to (5) of Hypothesis 1.1 are satisfied and if the data (u_0, u_1) belong to $\mathcal{A}_{m,s}$ for $s > 1$, then the energy solution to the Cauchy problem (1.2) satisfies the following estimates:

$$\|u(t,\cdot)\|_{L^2(\mathbb{R}^n)} \lesssim \big(1+B(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)} \|(u_0,u_1)\|_{\mathcal{A}_{m,s}}, \tag{1.16}$$

$$\|u_t(t,\cdot)\|_{L^2(\mathbb{R}^n)} \lesssim b(t)^{-1}\big(1+B(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-1} \|(u_0,u_1)\|_{\mathcal{A}_{m,s}}, \tag{1.17}$$

$$\||D|^s u(t,\cdot)\|_{L^2(\mathbb{R}^n)} \lesssim \big(1+B(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s}{2}} \|(u_0,u_1)\|_{\mathcal{A}_{m,s}}, \tag{1.18}$$

$$\||D|^{s-1} u_t(t,\cdot)\|_{L^2(\mathbb{R}^n)} \lesssim b(t)^{-1}\big(1+B(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s-1}{2}-1} \|(u_0,u_1)\|_{\mathcal{A}_{m,s}}. \tag{1.19}$$

Proof. To prove these statements we use the same ideas which are used in the proof of Theorem 1.2. Indeed, from [69] we have already the desired estimates for

$$\|u(t,\cdot)\|_{L^2(\mathbb{R}^n)},\ \ \|u_t(t,\cdot)\|_{L^2(\mathbb{R}^n)},\ \ \||D|^q u(t,\cdot)\|_{L^2(\mathbb{R}^n)} \ \text{ and for }\ \||D|^q u_t(t,\cdot)\|_{L^2(\mathbb{R}^n)},$$

where $q > 1$ is an integer number. To get (1.18) we use the interpolation theorem between

$$L : (u_0, u_1) \in \mathcal{A}_{m,1} \longrightarrow u \in \dot{H}^1(\mathbb{R}^n)$$

with

$$\|L\|_{\mathcal{A}_{m,1}\longrightarrow \dot{H}^1(\mathbb{R}^n)} \lesssim \big(1+B(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{1}{2}};$$

and

$$L : (u_0, u_1) \in \mathcal{A}_{m,q} \longrightarrow u \in \dot{H}^q(\mathbb{R}^n)$$

with

$$\|L\|_{\mathcal{A}_{m,q}\longrightarrow \dot{H}^q(\mathbb{R}^n)} \lesssim \big(1+B(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{q}{2}}.$$

To get (1.19) we use the interpolation theorem between

$$\tilde{L} : (u_0,u_1) \in \mathcal{A}_{m,1} \longrightarrow u_t \in L^2(\mathbb{R}^n)$$

with

$$\|\tilde{L}\|_{\mathcal{A}_{m,1}\longrightarrow L^2(\mathbb{R}^n)} \lesssim b(t)^{-1}\big(1+B(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-1};$$

and

$$\tilde{L} : (u_0,u_1) \in \mathcal{A}_{m,q} \longrightarrow u_t \in \dot{H}^q(\mathbb{R}^n)$$

with

$$\|\tilde{L}\|_{\mathcal{A}_{m,q}\longrightarrow \dot{H}^q(\mathbb{R}^n)} \lesssim b(t)^{-1}\big(1+B(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{q}{2}-1}.$$

In this way we complete the proof. □

In [4] the authors derived Matsumura-type estimates for solutions to the family of parameter-dependent Cauchy problems

$$v_{tt} - \Delta v + b(t)v_t = 0, \quad v(\tau,x) = 0, \;\; v_t(\tau,x) = g(\tau,x), \quad \tau \geq 0. \tag{1.20}$$

Theorem 1.4. *Let $b = b(t)$ satisfy Hypothesis 1.1 and let $g \in L^2(\mathbb{R}^n) \cap L^m(\mathbb{R}^n)$ for some $m \in [1,2)$. Then the energy solution $v = v(t,x)$ to (1.20) satisfies the following estimates:*

$$\big\|v(t,\cdot)\big\|_{L^2(\mathbb{R}^n)} \lesssim b(\tau)^{-1}\big(1+B(t,\tau)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)}\|g(\tau,\cdot)\|_{L^m(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)}, \tag{1.21}$$

$$\big\|\nabla v(t,\cdot)\big\|_{L^2(\mathbb{R}^n)} \lesssim b(\tau)^{-1}\big(1+B(t,\tau)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{1}{2}}\|g(\tau,\cdot)\|_{L^m(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)}, \tag{1.22}$$

$$\big\|v_t(t,\cdot)\big\|_{L^2(\mathbb{R}^n)} \lesssim b(t)^{-1}b(\tau)^{-1}\big(1+B(t,\tau)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-1}\|g(\tau,\cdot)\|_{L^m(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)}. \tag{1.23}$$

For the proof one can see [4].

From this theorem after using the interpolation argument from Theorem A.7 we can derive estimates for solutions to the family of parameter-dependent Cauchy problems (1.20) with low or large regular data.

Theorem 1.5. *(Low regular data)*
Let $b = b(t)$ satisfy Hypothesis 1.1 and let $g \in L^2(\mathbb{R}^n) \cap L^m(\mathbb{R}^n)$ for some $m \in [1,2)$. Then the Sobolev solution $v = v(t,x)$ to (1.20) satisfies for $s \in (0,1)$ the following estimates:

$$\big\|v(t,\cdot)\big\|_{L^2(\mathbb{R}^n)} \lesssim b(\tau)^{-1}\big(1+B(t,\tau)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)}\big\|g(\tau,\cdot)\big\|_{L^m(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)}, \tag{1.24}$$

$$\big\||D|^s v(t,\cdot)\big\|_{L^2(\mathbb{R}^n)} \lesssim b(\tau)^{-1}\big(1+B(t,\tau)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s}{2}}\big\|g(\tau,\cdot)\big\|_{L^m(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)}. \tag{1.25}$$

Theorem 1.6. *(High regular data)*
Let $b = b(t)$ satisfy Hypothesis 1.1 and let $g \in H^{s-1}(\mathbb{R}^n) \cap L^m(\mathbb{R}^n)$ for some $m \in [1, 2)$. Then the energy solution $v = v(t, x)$ to (1.20) satisfies for $s > 1$ the following estimates:

$$\begin{aligned}
\big\|v(t,\cdot)\big\|_{L^2(\mathbb{R}^n)} &\lesssim b(\tau)^{-1}\big(1+B(t,\tau)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)}\|g(\tau,\cdot)\|_{L^m(\mathbb{R}^n)\cap H^{s-1}(\mathbb{R}^n)},\\
\big\|v_t(t,\cdot)\big\|_{L^2(\mathbb{R}^n)} &\lesssim b(t)^{-1}b(\tau)^{-1}\big(1+B(t,\tau)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-1}\|g(\tau,\cdot)\|_{L^m(\mathbb{R}^n)\cap H^{s-1}(\mathbb{R}^n)},\\
\big\||D|^s v(t,\cdot)\big\|_{L^2(\mathbb{R}^n)} &\lesssim b(\tau)^{-1}\big(1+B(t,\tau)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s}{2}}\|g(\tau,\cdot)\|_{L^m(\mathbb{R}^n)\cap H^{s-1}(\mathbb{R}^n)}, \quad (1.26)
\end{aligned}$$

$$\big\||D|^{s-1}v_t(t,\cdot)\big\|_{L^2(\mathbb{R}^n)} \lesssim b(t)^{-1}b(\tau)^{-1}\big(1+B(t,\tau)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s-1}{2}-1}\|g(\tau,\cdot)\|_{L^m(\mathbb{R}^n)\cap H^{s-1}(\mathbb{R}^n)}. \quad (1.27)$$

Remark 1.1. *To prove the statements of Theorems 1.5 and 1.6 it is sufficient to use the estimate (7.1) from the paper [4] and an interpolation argument similar as we did in the proofs of the previous Theorems 1.2 and 1.3.*

1.2. Low regular data

To formulate the following results we introduce the exponents $p_{Fuj,m}(n) = 1 + \frac{2m}{n}$ for $m \in [1, 2)$ and $p_{GN,s}(n) = \frac{n}{n-2s}$.

Theorem 1.7. *Let $n \leq \frac{4s}{2-m}, n < \frac{2sm}{m-s}, s \in (0, 1)$ and $m \in [1, 2)$. The data (u_0, u_1) are supposed to belong to $\mathcal{A}_{m,s}$. Moreover, the exponent p satisfies*

$$p > p_{Fuj,m}(n), \quad (1.28)$$

and

$$\begin{array}{ccc}
\frac{2}{m} \leq p, s \in [\frac{1}{2}, 1) & \text{if} & n = 1,\\
\frac{2}{m} \leq p \leq p_{GN,s}(1), s \in (0, \frac{1}{2}) & \text{if} & n = 1,\\
\frac{2}{m} \leq p \leq p_{GN,s}(n) & \text{if} & n \geq 2.
\end{array} \quad (1.29)$$

Then, there exists a small constant ϵ_0 such that if

$$\|(u_0, u_1)\|_{\mathcal{A}_{m,s}} \leq \epsilon_0,$$

then there exists a uniquely determined globally (in time) Sobolev solution to the Cauchy problem

$$u_{tt} - \Delta u + b(t)u_t = |u|^p, \;\; u(0, x) = u_0(x), \;\; u_t(0, x) = u_1(x),$$

where $(t, x) \in [0, \infty) \times \mathbb{R}^n$.
This solution belongs to $\mathcal{C}\big([0, \infty), H^s(\mathbb{R}^n)\big)$. Furthermore, the solution satisfies the decay estimates

$$\begin{aligned}
&\|u(t,\cdot)\|_{L^2(\mathbb{R}^n)} \lesssim \big(1+B(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)}\|(u_0,u_1)\|_{\mathcal{A}_{m,s}},\\
&\||D|^s u(t,\cdot)\|_{L^2(\mathbb{R}^n)} \lesssim \big(1+B(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s}{2}}\|(u_0,u_1)\|_{\mathcal{A}_{m,s}}.
\end{aligned}$$

Remark 1.2. *The restriction for n depends heavily on the choice of the parameters m and s. For $n \geq 3$ we have the following statements:*

1. *If $m \in \left[\frac{-n+\sqrt{n^2+16n}}{4}, 2\right)$, then $\max\left\{\frac{2}{m}; p_{Fuj,m}(n)\right\} = p_{Fuj,m}(n)$ and the restriction of the dimension n will be $n < \frac{2sm}{m-s}$.*

2. *If $m \in \left[1, \frac{-n+\sqrt{n^2+16n}}{4}\right)$, then $\max\left\{\frac{2}{m}; p_{Fuj,m}(n)\right\} = \frac{2}{m}$ and the restriction of the dimension n will be $n \leq \frac{4s}{2-m}$.*

Example 1.2. *We discuss in the following table the conditions (1.28) and (1.29) in some special cases which depend on the parameters n, m and s to get the admissible range for p. We are interested in the case $n \leq 6$ only.*

n	m	Regularity s	Admissible range for p
$n=1$	$m \in [1,2)$	$s \in [\frac{1}{2}, 1)$	$1+2m < p < \infty$
		$s \in (\frac{m}{1+2m}, \frac{1}{2})$	$1+2m < p \leq \frac{1}{1-2s}$
$n=2$	$m \in [1,2)$	$s \in (\frac{m}{1+m}, 1)$	$1+m < p \leq \frac{1}{1-s}$
$n=3$	$m \in [\frac{-3+\sqrt{57}}{4}, 2)$	$s \in (\frac{3m}{2m+3}, 1)$	$1+\frac{2m}{3} < p \leq \frac{3}{3-2s}$
	$m \in [1, \frac{-3+\sqrt{57}}{4})$	$s \in [\frac{6-3m}{4}, 1)$	$\frac{2}{m} \leq p \leq \frac{3}{3-2s}$
$n=4$	$m \in [\frac{-2+\sqrt{20}}{2}, 2)$	$s \in (\frac{2m}{m+2}, 1)$	$1+\frac{m}{2} < p \leq \frac{2}{2-s}$
	$m \in (1, \frac{-2+\sqrt{20}}{2})$	$s \in [2-m, 1)$	$\frac{2}{m} \leq p \leq \frac{2}{2-s}$
$n=5$	$m \in [\frac{-5+\sqrt{105}}{4}, \frac{5}{3}]$	$s \in (\frac{5m}{2m+5}, 1)$	$1+\frac{2m}{5} < p \leq \frac{5}{5-2s}$
	$m \in [\frac{6}{5}, \frac{-5+\sqrt{105}}{4})$	$s \in [\frac{10-5m}{4}, 1)$	$\frac{2}{m} \leq p \leq \frac{5}{5-2s}$
$n=6$	$m \in [\frac{-3+\sqrt{33}}{2}, \frac{3}{2})$	$s \in (\frac{3m}{m+3}, 1)$	$1+\frac{m}{3} < p \leq \frac{3}{3-s}$
	$m \in [\frac{4}{3}, \frac{-3+\sqrt{33}}{2})$	$s \in [\frac{6-3m}{2}, 1)$	$\frac{2}{m} \leq p \leq \frac{3}{3-s}$

Example 1.3. *From the previous table we can construct several examples.*

1. *For $n = 3$, if we take $m = \frac{7}{4} \in \left[\frac{-3+\sqrt{57}}{4}, 2\right)$, then we have the admissible range for p as follows: $\frac{13}{6} < p \leq \frac{3}{3-2s}$ for $s \in \left(\frac{21}{26}, 1\right)$.*

2. *For $n = 6$, if we take $m = \frac{7}{5} \in [\frac{-3+\sqrt{33}}{2}, \frac{3}{2})$ implies $\max\left\{\frac{2}{m}; p_{Fuj,m}(6)\right\} = p_{Fuj,m}(6) = 1 + \frac{m}{3}$. Then we have the admissible range for p as follows: $\frac{22}{15} < p \leq \frac{3}{3-s}$ for $s \in \left(\frac{21}{22}, 1\right)$.*

Proof. We define the space of solutions $X(t)$ by

$$X(t) = \mathcal{C}([0,t], H^s(\mathbb{R}^n))$$

with the norm

$$\begin{aligned}
\|u\|_{X(t)} \;=\; \sup_{\tau\in[0,t]} \Big\{ & \big(1+B(\tau,0)\big)^{\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)} \|u(\tau,\cdot)\|_{L^2(\mathbb{R}^n)} \\
& + \big(1+B(\tau,0)\big)^{\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)+\frac{s}{2}} \||D|^s u(\tau,\cdot)\|_{L^2(\mathbb{R}^n)} \Big\}.
\end{aligned}$$

We remark that if $u \in X(t)$, then $\|u\|_{X(\tau)} \leq \|u\|_{X(t)}$ for any $0 \leq \tau \leq t$. We introduce the operator N by

$$N : u \in X(t) \to Nu = Nu(t,x) := u^{ln}(t,x) + u^{nl}(t,x). \tag{1.30}$$

We denote by $E_0(t,0,x)$ and $E_1(t,0,x)$ the fundamental solutions to the linear equation, namely

$$u^{ln}(t,x) := E_0(t,0,x) *_{(x)} u_0(x) + E_1(t,0,x) *_{(x)} u_1(x)$$

is a solution to the Cauchy problem

$$u_{tt} - \Delta u + b(t)u_t = 0, \;\; u(0,x) = u_0(x), \;\; u_t(0,x) = u_1(x),$$

and

$$u^{nl}(t,x) := \int_0^t E_1(t,\tau,x) *_{(x)} |u(\tau,x)|^p d\tau$$

is a solution to the Cauchy problem

$$u_{tt} - \Delta u + b(t)u_t = |u(t,x)|^p, \;\; u(0,x) = 0, \;\; u_t(0,x) = 0.$$

After Proposition A.8 it is sufficient to prove the following inequalities:

$$\begin{aligned} \|Nu\|_{X(t)} &\leq C_0(t)\|(u_0,u_1)\|_{\mathcal{A}_{m,s}} + C_1(t)\|u\|^p_{X(t)}, && (1.31)\\ \|Nu - Nv\|_{X(t)} &\leq C_2(t)\|u-v\|_{X(t)}\big(\|u\|^{p-1}_{X(t)} + \|v\|^{p-1}_{X(t)}\big), && (1.32) \end{aligned}$$

where $C_1(t), C_2(t) \longrightarrow 0$ for $t \longrightarrow 0$ and $C_1(t), C_2(t) \leq C$ for all $t \in [0,\infty)$. We begin by the proof of (1.31). From the estimates (1.13) and (1.14) of Theorem 1.2 and the definition of the norm of solutions space $X(t)$ we have

$$\begin{aligned} &\big\|u^{ln}\big\|_{X(t)} \\ &= \sup_{\tau\in[0,t]} \Big\{ \big(1+B(\tau,0)\big)^{\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)} \big\|u^{ln}(\tau,\cdot)\big\|_{L^2(\mathbb{R}^n)} \\ &\qquad + \big(1+B(\tau,0)\big)^{\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)+\frac{s}{2}} \big\||D|^s u^{ln}(\tau,\cdot)\big\|_{L^2(\mathbb{R}^n)} \Big\} \\ &\lesssim \sup_{\tau\in[0,t]} \Big\{ \big(1+B(\tau,0)\big)^{\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)} \big(1+B(\tau,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)} \big\|(u_0,u_1)\big\|_{\mathcal{A}_{m,s}} \\ &\qquad + \big(1+B(\tau,0)\big)^{\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)+\frac{s}{2}} \big(1+B(\tau,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s}{2}} \big\|(u_0,u_1)\big\|_{\mathcal{A}_{m,s}} \Big\} \\ &\lesssim \big\|(u_0,u_1)\big\|_{\mathcal{A}_{m,s}}. \end{aligned}$$

Consequently,

$$\big\|u^{ln}\big\|_{X(t)} \lesssim \|(u_0,u_1)\|_{\mathcal{A}_{m,s}}. \tag{1.33}$$

To prove (1.31) it is sufficient to prove the following inequality:

$$\big\|u^{nl}\big\|_{X(t)} \lesssim \|u\|^p_{X(t)}.$$

By choosing $m = 2$ for $\tau \in [\frac{t}{2}, t]$ and after using the estimate (1.25) from Theorem 1.5 we obtain

$$\begin{aligned}
&\big\||D|^s u^{nl}(t,\cdot)\big\|_{L^2(\mathbb{R}^n)}\\
&\lesssim \int_0^t b(\tau)^{-1}(1+B(t,\tau))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s}{2}}\big\||u(\tau,x)|^p\big\|_{L^m(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)}d\tau\\
&\lesssim \int_0^{\frac{t}{2}} b(\tau)^{-1}(1+B(t,\tau))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s}{2}}\||u(\tau,x)|^p\|_{L^m(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)}d\tau\\
&\qquad+\int_{\frac{t}{2}}^t b(\tau)^{-1}(1+B(t,\tau))^{-\frac{s}{2}}\||u(\tau,x)|^p\|_{L^2(\mathbb{R}^n)}d\tau.
\end{aligned}$$

We have

$$\||u(\tau,x)|^p\|_{L^m(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)} \lesssim \||u(\tau,x)|^p\|_{L^m(\mathbb{R}^n)} + \||u(\tau,x)|^p\|_{L^2(\mathbb{R}^n)}.$$

By using the Gagliardo-Nirenberg inequality from Corollary A.1 we estimate both terms of the right-hand side. In this way we obtain for the fist term

$$\begin{aligned}
\big\||u(\tau,x)|^p\big\|_{L^m(\mathbb{R}^n)} &= \Big(\int_{\mathbb{R}^n}|u(\tau,x)|^{mp}dx\Big)^{\frac{1}{mp}p}\\
&= \|u(\tau,\cdot)\|^p_{L^{mp}(\mathbb{R}^n)}\\
&\lesssim \|u(\tau,\cdot)\|^{(1-\theta)p}_{L^2(\mathbb{R}^n)}\,\||D|^s u(\tau,\cdot)\|^{\theta p}_{L^2(\mathbb{R}^n)},
\end{aligned}$$

where

$$\theta = \frac{n}{s}\Big(\frac{1}{2}-\frac{1}{mp}\Big) \in [0,1]$$

due to the condition (1.29) for p. By using the norm of solution space $X(t)$ for $0 \le \tau \le t$ we get

$$\|u(\tau,\cdot)\|^p_{L^{mp}(\mathbb{R}^n)} \lesssim (1+B(\tau,0))^{(1-\theta)p\left(-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)\right)+\theta p\left(-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s}{2}\right)}\|u\|^p_{X(t)}.$$

Then

$$\|u(\tau,\cdot)\|^p_{L^{mp}(\mathbb{R}^n)} \lesssim (1+B(\tau,0))^{-\frac{n}{2m}p+\frac{n}{2m}}\|u\|^p_{X(t)}. \tag{1.34}$$

The same ideas are used to estimate $\||u(\tau,x)|^p\|_{L^2}$. By using the classical Gagliardo-Nirenberg inequality and the definition of the norm of solution space $X(t)$ we get

$$\begin{aligned}
\||u(\tau,x)|^p\|_{L^2(\mathbb{R}^n)} &= \Big(\int_{\mathbb{R}^n}|u(\tau,x)|^{2p}dx\Big)^{\frac{1}{2p}p}\\
&= \|u(\tau,\cdot)\|^p_{L^{2p}(\mathbb{R}^n)}\\
&\lesssim \|u(\tau,\cdot)\|^{p(1-\tilde{\theta})}_{L^2(\mathbb{R}^n)}\big\||D|^s u(\tau,\cdot)\big\|^{p\tilde{\theta}}_{L^2(\mathbb{R}^n)}\\
&\lesssim (1+B(\tau,0))^{(1-\tilde{\theta})p\left(-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)\right)+\tilde{\theta}p\left(-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s}{2}\right)}\|u\|^p_{X(t)}\\
&\lesssim (1+B(t,\tau))^{-\frac{s}{2}}(1+B(\tau,0))^{-\frac{n}{2m}p+\frac{n}{4}}\|u\|^p_{X(t)},
\end{aligned}$$

where

$$\tilde{\theta} = \frac{n}{s}\Big(\frac{1}{2}-\frac{1}{2p}\Big) \in [0,1]$$

from condition (1.29) for p. Hence, we may conclude for $0 \leq \tau \leq t$ the following estimate:

$$\|u(\tau,\cdot)\|^p_{L^{2p}(\mathbb{R}^n)} \lesssim (1+B(\tau,0))^{-\frac{n}{2m}p+\frac{n}{4}}\|u\|^p_{X(t)}. \tag{1.35}$$

We estimate the integral over $[0,\frac{t}{2}]$ by using (1.34), (1.35) and (1.6) as follows:

$$\begin{aligned}
&\int_0^{\frac{t}{2}} b(\tau)^{-1}(1+B(t,\tau))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s}{2}}\||u(\tau,x)|^p\|_{L^m(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)}d\tau\\
&\quad\lesssim \|u\|^p_{X(t)}\int_0^{\frac{t}{2}} b(\tau)^{-1}(1+B(t,\tau))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s}{2}}(1+B(\tau,0))^{-\frac{n}{2m}p+\frac{n}{2m}}d\tau\\
&\quad\lesssim \|u\|^p_{X(t)}(1+B(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s}{2}}\int_0^{\frac{t}{2}} b(\tau)^{-1}(1+B(\tau,0))^{-\frac{n}{2m}p+\frac{n}{2m}}d\tau\\
&\quad\lesssim \|u\|^p_{X(t)}(1+B(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s}{2}}
\end{aligned}$$

for $p > p_{Fuj,m}(n)$ which is exactly condition (1.28).
For the integral over $[\frac{t}{2},t]$ we use (1.35), (1.7) and (1.9) to get

$$\begin{aligned}
&\int_{\frac{t}{2}}^t b(\tau)^{-1}(1+B(t,\tau))^{-\frac{s}{2}}\big\||u(\tau,x)|^p\big\|_{L^2(\mathbb{R}^n)}d\tau\\
&\quad\lesssim \|u\|^p_{X(t)}\int_{\frac{t}{2}}^t b(\tau)^{-1}(1+B(t,\tau))^{-\frac{s}{2}}(1+B(\tau,0))^{-\frac{n}{2m}p+\frac{n}{4}}d\tau\\
&\quad\lesssim \|u\|^p_{X(t)}(1+B(t,0))^{-\frac{n}{2m}p+\frac{n}{4}}(1+B(t,0))^{1-\frac{s}{2}}\\
&\quad\lesssim \|u\|^p_{X(t)}(1+B(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s}{2}}
\end{aligned}$$

for $p > p_{Fuj,m}(n)$. Summarizing, we arrive at the estimate

$$\big\||D|^s u^{nl}(t,\cdot)\big\|_{L^2(\mathbb{R}^n)} \lesssim \|u\|^p_{X(t)}(1+B(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s}{2}}. \tag{1.36}$$

In the same way one can derive

$$\big\|u^{nl}(t,\cdot)\big\|_{L^2(\mathbb{R}^n)} \lesssim \|u\|^p_{X(t)}(1+B(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)}. \tag{1.37}$$

Using the norm of solution space $X(t)$ and (1.36), (1.37) we get

$$\begin{aligned}
&\|u^{nl}\|_{X(t)}\\
&\quad=\sup_{\tau\in[0,t]}\Big\{\big(1+B(\tau,0)\big)^{\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)}\big\|u^{nl}(\tau,\cdot)\big\|_{L^2(\mathbb{R}^n)}\\
&\qquad\qquad+\big(1+B(\tau,0)\big)^{\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)+\frac{s}{2}}\big\||D|^s u^{nl}(\tau,\cdot)\big\|_{L^2(\mathbb{R}^n)}\Big\}\\
&\quad\lesssim\sup_{\tau\in[0,t]}\Big\{\big(1+B(\tau,0)\big)^{\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)}\big(1+B(\tau,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)}\|u\|^p_{X(\tau)}\\
&\qquad\qquad+\big(1+B(\tau,0)\big)^{\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)+\frac{s}{2}}\big(1+B(\tau,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s}{2}}\|u\|^p_{X(\tau)}\Big\}\\
&\quad\lesssim \|u\|^p_{X(t)}.
\end{aligned}$$

So, it follows the desired estimate

$$\big\|u^{nl}\big\|_{X(t)} \lesssim \|u\|^p_{X(t)}. \tag{1.38}$$

Taking into consideration (1.33) and (1.38) the estimate (1.31) is proved.
Now we turn to (1.32). We assume that u and v are two functions belonging to $X(t)$. Then we have

$$\begin{aligned}
&\big\||D|^s(Nu-Nv)(t,\cdot)\big\|_{L^2(\mathbb{R}^n)}\\
&\quad= \Big\||D|^s\int_0^t E_1(t,\tau,x)*_{(x)}\big(|u(\tau,x)|^p-|v(\tau,x)|^p\big)d\tau\Big\|_{L^2(\mathbb{R}^n)}\\
&\quad\lesssim \int_0^{\frac{t}{2}} b(\tau)^{-1}(1+B(t,\tau))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s}{2}}\\
&\qquad\quad\times\big\|\big(|u(\tau,x)|^p-|v(\tau,x)|^p\big)\big\|_{L^m(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)}\,d\tau\\
&\qquad\quad+\int_{\frac{t}{2}}^t b(\tau)^{-1}(1+B(t,\tau))^{-\frac{s}{2}}\big\|\big(|u(\tau,x)|^p-|v(\tau,x)|^p\big)\big\|_{L^2(\mathbb{R}^n)}\,d\tau.
\end{aligned} \tag{1.39}$$

Hölder's inequality implies

$$\begin{aligned}
&\big\||u(\tau,x)|^p-|v(\tau,x)|^p\big\|_{L^2(\mathbb{R}^n)}\\
&\qquad\lesssim \big\|u(\tau,\cdot)-v(\tau,\cdot)\big\|_{L^{2p}(\mathbb{R}^n)}\big(\|u(\tau,\cdot)\|^{p-1}_{L^{2p}(\mathbb{R}^n)}+\|v(\tau,\cdot)\|^{p-1}_{L^{2p}(\mathbb{R}^n)}\big),
\end{aligned} \tag{1.40}$$

$$\begin{aligned}
&\big\||u(\tau,x)|^p-|v(\tau,x)|^p\big\|_{L^m(\mathbb{R}^n)}\\
&\qquad\lesssim \big\|u(\tau,\cdot)-v(\tau,\cdot)\big\|_{L^{mp}(\mathbb{R}^n)}\big(\|u(\tau,\cdot)\|^{p-1}_{L^{mp}(\mathbb{R}^n)}+\|v(\tau,\cdot)\|^{p-1}_{L^{mp}(\mathbb{R}^n)}\big).
\end{aligned} \tag{1.41}$$

By using the norm of solution space $X(t)$ and after applying the classical Gagliardo-Nirenberg inequality as we did for (1.34) and (1.35) we obtain for $0\le\tau\le t$ the following estimates:

$$\|u(\tau,\cdot)-v(\tau,\cdot)\|_{L^{2p}(\mathbb{R}^n)}\lesssim(1+B(\tau,0))^{-\frac{n}{2m}+\frac{n}{4p}}\|u-v\|_{X(t)},$$

$$\|u(\tau,\cdot)\|^{p-1}_{L^{2p}(\mathbb{R}^n)}\lesssim(1+B(\tau,0))^{\left(-\frac{n}{2m}+\frac{n}{4p}\right)(p-1)}\|u\|^{p-1}_{X(t)},$$

$$\|v(\tau,\cdot)\|^{p-1}_{L^{2p}(\mathbb{R}^n)}\lesssim(1+B(\tau,0))^{\left(-\frac{n}{2m}+\frac{n}{4p}\right)(p-1)}\|v\|^{p-1}_{X(t)},$$

$$\|u(\tau,\cdot)-v(\tau,\cdot)\|_{L^{mp}(\mathbb{R}^n)}\lesssim(1+B(\tau,0))^{-\frac{n}{2m}+\frac{n}{2mp}}\|u-v\|_{X(t)},$$

$$\|u(\tau,\cdot)\|^{p-1}_{L^{mp}(\mathbb{R}^n)}\lesssim(1+B(\tau,0))^{\left(-\frac{n}{2m}+\frac{n}{2mp}\right)(p-1)}\|u\|^{p-1}_{X(t)},$$

$$\big\|v(\tau,\cdot)\big\|^{p-1}_{L^{mp}(\mathbb{R}^n)}\lesssim(1+B(\tau,0))^{\left(-\frac{n}{2m}+\frac{n}{2mp}\right)(p-1)}\|v\|^{p-1}_{X(t)}.$$

Then we get

$$\begin{aligned}
&\big\||u(\tau,x)|^p-|v(\tau,x)|^p\big\|_{L^2(\mathbb{R}^n)}\\
&\qquad\lesssim(1+B(\tau,0))^{-\frac{n}{2m}p+\frac{n}{4}}\|u-v\|_{X(t)}\big(\|u\|^{p-1}_{X(t)}+\|v\|^{p-1}_{X(t)}\big),
\end{aligned} \tag{1.42}$$

$$\begin{aligned}
&\big\||u(\tau,x)|^p-|v(\tau,x)|^p\big\|_{L^m(\mathbb{R}^n)}\\
&\qquad\lesssim(1+B(\tau,0))^{-\frac{n}{2m}p+\frac{n}{2m}}\|u-v\|_{X(t)}\big(\|u\|^{p-1}_{X(t)}+\|v\|^{p-1}_{X(t)}\big).
\end{aligned} \tag{1.43}$$

Applying the same ideas as we did to estimate $\big\||D|^s u^{nl}(t,\cdot)\big\|_{L^2(\mathbb{R}^n)}$, this means, after plugging (1.42) and (1.43) into (1.39) one can get for $p>p_{Fuj,m}(n)$ the following estimates:

$$\begin{aligned}
&\big\||D|^s(Nu-Nv)(t,\cdot)\big\|_{L^2(\mathbb{R}^n)}\\
&\qquad\lesssim(1+B(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s}{2}}\|u-v\|_{X(t)}\big(\|u\|^{p-1}_{X(t)}+\|v\|^{p-1}_{X(t)}\big),
\end{aligned}$$

$$\big\|(Nu-Nv)(t,\cdot)\big\|_{L^2(\mathbb{R}^n)} \lesssim (1+B(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)}\|u-v\|_{X(t)}\big(\|u\|^{p-1}_{X(t)}+\|v\|^{p-1}_{X(t)}\big).$$

Then from the definition of $X(t)$, the proof of (1.32) is completed. □

1.3. Data from the energy space

Let us come back to the Cauchy problem

$$u_{tt}-\Delta u+b(t)u_t=|u|^p,\;\; u(0,x)=u_0(x),\;\; u_t(0,x)=u_1(x), \tag{1.44}$$

where $(t,x)\in[0,\infty)\times\mathbb{R}^n$. If we assume in Theorem 1.7 that s tends to 1 with additional regularity $L^1(\mathbb{R}^n)$ for the data, this means $m=1$, then we obtain the global existence in time of Sobolev solutions in $C([0,\infty),H^1(\mathbb{R}^n))$. This coincides with the result in [4], where the data space is $(H^1(\mathbb{R}^n)\cap L^1(\mathbb{R}^n))\times(L^2(\mathbb{R}^n)\cap L^1(\mathbb{R}^n))$. But there the authors proved even the existence of energy solutions.

Theorem 1.8. *Let $n\le 4$ and $(u_0,u_1)\in\mathcal{A}_{1,1}$. The exponent p satisfies*

$$\begin{array}{rl} p>p_{Fuj}(n) & \text{if}\;\; n=1,2,\\ 2\le p\le 3=p_{GN}(3) & \text{if}\;\; n=3,\\ p=2=p_{GN}(4) & \text{if}\;\; n=4, \end{array}$$

where $p_{Fuj}(n)=1+\frac{2}{n}$ and $p_{GN}(n)=\frac{n}{n-2}$. Then, there exists a small constant ϵ_0 such that, if

$$\|(u_0,u_1)\|_{\mathcal{A}_{1,1}}\le\epsilon_0,$$

then there exists a uniquely determined globally (in time) energy solution to (1.44) in

$$\mathcal{C}([0,\infty),H^1(\mathbb{R}^n))\cap\mathcal{C}^1([0,\infty),L^2(\mathbb{R}^n)).$$

Furthermore, the solution satisfies the decay estimates:

$$\begin{aligned} \|u(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \big(1+B(t,0)\big)^{-\frac{n}{4}}\|(u_0,u_1)\|_{\mathcal{A}_{1,1}},\\ \|\nabla u(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \big(1+B(t,0)\big)^{-\frac{n}{4}-\frac{1}{2}}\|(u_0,u_1)\|_{\mathcal{A}_{1,1}},\\ \|u_t(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim b(t)^{-1}\big(1+B(t,0)\big)^{-\frac{n}{4}-1}\|(u_0,u_1)\|_{\mathcal{A}_{1,1}}. \end{aligned}$$

For the proof see [4].

If we suppose for the data an additional regularity $L^m(\mathbb{R}^n)$ with $m\in[1,2)$, then we get the following result which also can be found in the paper [5].

Theorem 1.9. *Let $n\le\frac{4}{2-m}$ and $n<\frac{2m}{m-1}$. The data (u_0,u_1) are assumed to belong to $\mathcal{A}_{m,1}$ for $m\in[1,2)$. The exponent p satisfies*

$$p>p_{Fuj,m}(n),\qquad \begin{array}{rl} \frac{2}{m}\le p<\infty & \text{if}\;\; n=1,2,\\ \frac{2}{m}\le p\le p_{GN}(n) & \text{if}\;\; 2<n\le 6. \end{array} \tag{1.45}$$

Then, there exists a small constant ϵ_0 such that, if

$$\|(u_0,u_1)\|_{\mathcal{A}_{m,1}} \le \epsilon_0,$$

then there exists a uniquely determined globally (in time) energy solution to (1.44) in

$$\mathcal{C}([0,\infty),H^1(\mathbb{R}^n))\cap \mathcal{C}^1([0,\infty),L^2(\mathbb{R}^n)).$$

Furthermore, the solution satisfies the decay estimates:

$$\begin{aligned}
\|u(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \big(1+B(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)}\|(u_0,u_1)\|_{\mathcal{A}_{m,1}},\\
\|\nabla u(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \big(1+B(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{1}{2}}\|(u_0,u_1)\|_{\mathcal{A}_{m,1}},\\
\|u_t(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim b(t)^{-1}\big(1+B(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-1}\|(u_0,u_1)\|_{\mathcal{A}_{m,1}}.
\end{aligned}$$

Remark 1.3. *In the last theorem we have results for $n \le 6$ only. For the case $n=7$ the admissible range for p will be empty as we will show later in detail. We will devote to the cases $n=6$ and $n=7$ only.*

__*The case $n=6$:*__ *If $m \in \big[\frac{-6+\sqrt{132}}{4},2\big)$, then $\max\big\{\frac{2}{m};p_{Fuj,m}(n)\big\}=p_{Fuj,m}(n)$ and $\frac{m}{3}+1<p\le\frac{3}{2}$ for $m\in\big[1,\frac{3}{2}\big)$. Finally, an admissible range for p exists only if $m\in\big[\frac{-6+\sqrt{132}}{4},2\big)\cap\big[1,\frac{3}{2}\big)=\big[\frac{-6+\sqrt{132}}{4},\frac{3}{2}\big)$.*
If $m\in\big[1,\frac{-6+\sqrt{132}}{4}\big)$, then $\max\big\{\frac{2}{m};p_{Fuj,m}(n)\big\}=\frac{2}{m}$ and $\frac{2}{m}\le p\le\frac{3}{2}$ for $m\in\big[\frac{4}{3},2\big)$. Finally, an admissible range for p exists only if $m\in\big[1,\frac{-6+\sqrt{132}}{4}\big)\cap\big[\frac{4}{3},2\big)=\big[\frac{4}{3},\frac{-6+\sqrt{132}}{4}\big)$.

__*The case $n=7$:*__ *If $m\in\big[\frac{-7+\sqrt{161}}{4},2\big)$, then $\max\big\{\frac{2}{m};p_{Fuj,m}(n)\big\}=p_{Fuj,m}(n)$ and $\frac{2m}{7}+1<p\le\frac{7}{5}$ for $m\in\big[1,\frac{7}{5}\big)$. Finally, an admissible range for p exists only if $m\in\big[\frac{-7+\sqrt{161}}{4},2\big)\cap\big[1,\frac{7}{5}\big)=\varnothing$.*
If $m\in\big[1,\frac{-7+\sqrt{161}}{4}\big)$, then $\max\big\{\frac{2}{m};p_{Fuj,m}(n)\big\}=\frac{2}{m}$ and $\frac{2}{m}\le p\le\frac{7}{5}$ for $m\in\big[\frac{10}{7},2\big)$. Finally, an admissible range for p exists only if $m\in\big[1,\frac{-7+\sqrt{161}}{4}\big]\cap\big[\frac{10}{7},2\big)=\varnothing$.

Remark 1.4. *Comparing the statements of Theorems 1.8 and 1.9 we deduce the benefit of the additional regularity $L^m(\mathbb{R}^n)$ of the data. It allows to have results for dimensions up to $n=6$ unlike the previous case of $L^1(\mathbb{R}^n)$ additional regularity where results are proved for $n\le 4$ only.*

Example 1.4. *Let us choose $b(t)=\frac{1}{(1+t)^r}$, where $r\in(-1,1)$. Then due to Theorem 1.9 we obtain the global existence (in time) of energy solutions to (1.44) for small initial data $(u_0,u_1)\in\mathcal{A}_{m,1}$. The solution satisfies the following estimates:*

$$\begin{aligned}
\|u(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim (1+t)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)(1+r)}\|(u_0,u_1)\|_{\mathcal{A}_{m,1}},\\
\|\nabla u(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim (1+t)^{-\left(\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)+\frac{1}{2}\right)(1+r)}\|(u_0,u_1)\|_{\mathcal{A}_{m,1}},\\
\|u_t(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim (1+t)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)(1+r)-1}\|(u_0,u_1)\|_{\mathcal{A}_{m,1}}.
\end{aligned}$$

To get the last estimate we use property (1.8).

Proof. We define the space of solutions $X(t)$ by

$$X(t) = \mathcal{C}([0,t], H^1(\mathbb{R}^n)) \cap \mathcal{C}^1([0,t], L^2(\mathbb{R}^n))$$

with the norm

$$\begin{aligned}\|u\|_{X(t)} = \sup_{\tau\in[0,t]} \Big\{ &\big(1+B(\tau,0)\big)^{\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)} \|u(\tau,\cdot)\|_{L^2(\mathbb{R}^n)} \\ &+\big(1+B(\tau,0)\big)^{\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)+\frac{1}{2}} \|\nabla u(\tau,\cdot)\|_{L^2(\mathbb{R}^n)} \\ &+b(\tau)\big(1+B(\tau,0)\big)^{\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)+1} \|u_t(\tau,\cdot)\|_{L^2(\mathbb{R}^n)} \Big\}.\end{aligned}$$

Let N be the operator which is defined by (1.30). So,

$$Nu(t,x) = E_0(t,0,x) *_{(x)} u_0(x) + E_1(t,0,x) *_{(x)} u_1(x) + \int_0^t E_1(t,\tau,x) *_{(x)} |u(\tau,x)|^p d\tau.$$

Our goal is to prove the inequalities of Proposition A.8. These inequalities are

$$\|Nu\|_{X(t)} \lesssim \|(u_0,u_1)\|_{\mathcal{A}_{m,1}} + \|u\|^p_{X(t)}, \tag{1.46}$$

$$\|Nu - Nv\|_{X(t)} \lesssim \|u-v\|_{X(t)}\big(\|u\|^{p-1}_{X(t)} + \|v\|^{p-1}_{X(t)}\big). \tag{1.47}$$

Let us start with inequality (1.46). Using the estimates (1.10), (1.11) and (1.12) of Theorem 1.1 and taking account of the norm of the solution space $X(t)$ we obtain immediately

$$\big\|u^{ln}\big\|_{X(t)} \lesssim \|(u_0,u_1)\|_{\mathcal{A}_{m,1}}. \tag{1.48}$$

To complete the proof of (1.46) we have to estimate $\|u^{nl}\|_{X(t)}$. After using the estimate (1.22) of Theorem 1.4 we get

$$\big\|\nabla u^{nl}(t,\cdot)\big\|_{L^2(\mathbb{R}^n)} \lesssim \int_0^t b(\tau)^{-1}(1+B(t,\tau))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{1}{2}} \||u(\tau,x)|^p\|_{L^m(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)} d\tau.$$

By choosing $m=2$ for the integral over $[\frac{t}{2},t]$ we get

$$\begin{aligned}\big\|\nabla u^{nl}(t,\cdot)\big\|_{L^2(\mathbb{R}^n)} \quad &\lesssim \int_0^{\frac{t}{2}} b(\tau)^{-1}(1+B(t,\tau))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{1}{2}} \||u(\tau,x)|^p\|_{L^m(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)} d\tau \\ &\quad + \int_{\frac{t}{2}}^t b(\tau)^{-1}(1+B(t,\tau))^{-\frac{1}{2}} \||u(\tau,x)|^p\|_{L^2(\mathbb{R}^n)} d\tau.\end{aligned}$$

Using the classical Gagliardo-Nirenberg inequality gives for $0 \le \tau \le t$ the estimate

$$\|u(\tau,\cdot)\|^p_{L^{mp}(\mathbb{R}^n)} \lesssim (1+B(\tau,0))^{-\frac{n}{2m}p+\frac{n}{2m}} \|u\|^p_{X(t)},$$

where $\theta = n\big(\frac{1}{2}-\frac{1}{mp}\big) \in [0,1]$ due to the conditions (1.45) for p.
Analogously,

$$\|u(\tau,\cdot)\|^p_{L^{2p}(\mathbb{R}^n)} \lesssim (1+B(\tau,0))^{-\frac{n}{2m}p+\frac{n}{4}} \|u\|^p_{X(t)},$$

due to the conditions (1.45) for p. Using these estimates and (1.6), (1.7) we may conclude for $p > p_{Fuj,m}(n)$ the estimates

$$\int_0^{\frac{t}{2}} b(\tau)^{-1}(1+B(t,\tau))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{1}{2}} \| |u(\tau,x)|^p \|_{L^m(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)} d\tau$$
$$\lesssim \|u\|^p_{X(t)}(1+B(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{1}{2}}$$

and

$$\int_{\frac{t}{2}}^{t} b(\tau)^{-1}(1+B(t,\tau))^{-\frac{1}{2}} \| |u(\tau,x)|^p \|_{L^2(\mathbb{R}^n)} d\tau \lesssim \|u\|^p_{X(t)}(1+B(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{1}{2}}.$$

Consequently,

$$\left\|\nabla u^{nl}(t,\cdot)\right\|_{L^2(\mathbb{R}^n)} \lesssim \|u\|^p_{X(t)}(1+B(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{1}{2}}. \tag{1.49}$$

Using the same ideas one can derive

$$\left\|u_t^{nl}(t,\cdot)\right\|_{L^2(\mathbb{R}^n)} \lesssim \|u\|^p_{X(t)} b(t)^{-1}(1+B(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-1}, \tag{1.50}$$
$$\|u^{nl}(t,\cdot)\|_{L^2(\mathbb{R}^n)} \lesssim \|u\|^p_{X(t)}(1+B(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)}. \tag{1.51}$$

Replacing the estimates (1.49), (1.50) and (1.51) in the norm of the solution space $X(t)$ we obtain

$$\left\|u^{nl}\right\|_{X(t)} \lesssim \|u\|^p_{X(t)}. \tag{1.52}$$

Then after the estimates (1.48) and (1.52) the proof of (1.46) is completed.
Now let us prove the second inequality (1.47). Using the estimate (1.22) of Theorem 1.4 with $m = 2$ for the integral over $[\frac{t}{2}, t]$ we obtain

$$\begin{aligned}
&\left\|\nabla(Nu-Nv)(t,\cdot)\right\|_{L^2(\mathbb{R}^n)} \\
&\quad = \left\|\nabla \int_0^t E_1(t,\tau,x) *_{(x)} \left(|u(\tau,x)|^p - |v(\tau,x)|^p\right) d\tau\right\|_{L^2(\mathbb{R}^n)} \\
&\quad \lesssim \int_0^{\frac{t}{2}} (1+B(t,\tau))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{1}{2}} \left\| \left(|u(\tau,x)|^p - |v(\tau,x)|^p\right) \right\|_{L^m(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)} d\tau \\
&\qquad + \int_{\frac{t}{2}}^t (1+B(t,\tau))^{-\frac{1}{2}} \left\|\left(|u(\tau,x)|^p - |v(\tau,x)|^p\right)\right\|_{L^2(\mathbb{R}^n)} d\tau.
\end{aligned}$$

After using the estimates (1.42) and (1.43) of the proof of Theorem 1.7 and (1.6), (1.7) we obtain in the same way we estimated $\left\|\nabla u^{nl}(t,\cdot)\right\|_{L^2(\mathbb{R}^n)}$ under condition (1.9) the following estimate:

$$\begin{aligned}
\left\|\nabla(Nu-Nv)(t,\cdot)\right\|_{L^2(\mathbb{R}^n)} &\lesssim (1+B(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{1}{2}} \|u-v\|_{X(t)}\left(\|u\|^{p-1}_{X(t)} + \|v\|^{p-1}_{X(t)}\right), \\
\left\|(Nu-Nv)(t,\cdot)\right\|_{L^2(\mathbb{R}^n)} &\lesssim (1+B(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)} \|u-v\|_{X(t)}\left(\|u\|^{p-1}_{X(t)} + \|v\|^{p-1}_{X(t)}\right), \\
\left\|\partial_t(Nu-Nv)(t,\cdot)\right\|_{L^2(\mathbb{R}^n)} &\lesssim b(t)^{-1}(1+B(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-1} \\
&\qquad \times \|u-v\|_{X(t)}\left(\|u\|^{p-1}_{X(t)} + \|v\|^{p-1}_{X(t)}\right).
\end{aligned}$$

Then from the definition of the norm of $X(t)$ the proof is completed. $\square$

1.4. Data from Sobolev spaces with suitable regularity

Again we turn to the Cauchy problem

$$u_{tt} - \Delta u + b(t)u_t = |u|^p, \;\; (u, u_t)(0, x) = (u_0, u_1)(x). \tag{1.53}$$

But now we assume that the data have a suitable larger regularity, i.e.,

$$(u_0, u_1) \in H^s(\mathbb{R}^n) \times H^{s-1}(\mathbb{R}^n), s \in \Big(1, 1 + \frac{n}{2}\Big]$$

with an additional regularity $L^m(\mathbb{R}^n)$, $m \in [1, 2)$. In this section we shall use a generalized (fractional) Gagliardo-Nirenberg inequality used in the papers [19] and [55]. Furthermore, we shall use a fractional Leibniz rule and a fractional chain rule which are explained in Propositions A.2 and A.4 from the Appendix.

Theorem 1.10. *Let $n \geq 4$ and $s \in [3, \frac{n}{2} + 1]$. The data (u_0, u_1) are supposed to belong to $\mathcal{A}_{m,s}$, where $m \in [1, 2)$. Finally, the following conditions are satisfied for the exponent p:*

$$\begin{array}{llll} \lceil s \rceil < & p & \text{if} & s \in [\frac{n}{2}, \frac{n}{2} + 1], \\ \lceil s \rceil < & p \leq 1 + \frac{2}{n-2s} & \text{if} & s \in [3, \frac{n}{2}). \end{array} \tag{1.54}$$

Then, there exists a small constant ϵ_0 such that if

$$\|(u_0, u_1)\|_{\mathcal{A}_{m,s}} \leq \epsilon_0,$$

then there exists a uniquely determined globally (in time) energy solution to (1.53) in

$$\mathcal{C}\big([0, \infty), H^s(\mathbb{R}^n)\big) \cap \mathcal{C}^1\big([0, \infty), H^{s-1}(\mathbb{R}^n)\big).$$

Furthermore, the solution satisfies the estimates:

$$\begin{aligned}
\|u(t, \cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \big(1 + B(t, 0)\big)^{-\frac{n}{2}\left(\frac{1}{m} - \frac{1}{2}\right)} \|(u_0, u_1)\|_{\mathcal{A}_{m,s}}, \\
\|u_t(t, \cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim b(t)^{-1}\big(1 + B(t, 0)\big)^{-\frac{n}{2}\left(\frac{1}{m} - \frac{1}{2}\right) - 1} \|(u_0, u_1)\|_{\mathcal{A}_{m,s}}, \\
\||D|^s u(t, \cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \big(1 + B(t, 0)\big)^{-\frac{n}{2}\left(\frac{1}{m} - \frac{1}{2}\right) - \frac{s}{2}} \|(u_0, u_1)\|_{\mathcal{A}_{m,s}}, \\
\||D|^{s-1} u_t(t, \cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim b(t)^{-1}\big(1 + B(t, 0)\big)^{-\frac{n}{2}\left(\frac{1}{m} - \frac{1}{2}\right) - 1 - \frac{s-1}{2}} \|(u_0, u_1)\|_{\mathcal{A}_{m,s}}.
\end{aligned}$$

Example 1.5. *We discuss in the following table the condition (1.54) in some special cases for $m \in [1, 2)$, n and s to get the admissible range for p.*

dimension n	regularity s	admissible range for p
$n = 4$	$s = 3$	$p > 3$
$n = 5$	$s \in \left[3, \frac{7}{2}\right]$	$p > \lceil s \rceil$
$n = 6$	$s \in [3, 4]$	$p > \lceil s \rceil$

Proof. We define the space of solutions $X(t)$ by

$$X(t) = \mathcal{C}([0,t], H^s(\mathbb{R}^n)) \cap \mathcal{C}^1([0,t], H^{s-1}(\mathbb{R}^n))$$

with the norm

$$\begin{aligned}
\|u\|_{X(t)} = \sup_{\tau\in[0,t]} \Big\{ & \big(1+B(\tau,0)\big)^{\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)} \|u(\tau,\cdot)\|_{L^2(\mathbb{R}^n)} \\
& + \big(1+B(\tau,0)\big)^{\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)+\frac{s}{2}} \||D|^s u(\tau,\cdot)\|_{L^2(\mathbb{R}^n)} \\
& + b(\tau)\big(1+B(\tau,0)\big)^{\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)+1} \|u_t(\tau,\cdot)\|_{L^2(\mathbb{R}^n)} \\
& + b(\tau)\big(1+B(\tau,0)\big)^{\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)+\frac{s-1}{2}+1} \||D|^{s-1} u_t(\tau,\cdot)\|_{L^2(\mathbb{R}^n)} \Big\}.
\end{aligned}$$

Let N be the operator which is defined in (1.30). Our aim is to prove the inequalities

$$\|Nu\|_{X(t)} \lesssim \|(u_0,u_1)\|_{\mathcal{A}_{m,s}} + \|u\|_{X(t)}^p, \tag{1.55}$$

$$\|Nu - Nv\|_{X(t)} \lesssim \|u-v\|_{X(t)}\big(\|u\|_{X(t)}^{p-1} + \|v\|_{X(t)}^{p-1}\big). \tag{1.56}$$

We begin the proof of inequality (1.55). We have for the norm $\|u^{ln}\|_{X(t)}$ of the linear part u^{ln} of the solution u

$$\begin{aligned}
& \|u^{ln}\|_{X(t)} \\
& = \sup_{\tau\in[0,t]} \Big\{ \big(1+B(\tau,0)\big)^{\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)} \|u^{ln}(\tau,\cdot)\|_{L^2(\mathbb{R}^n)} \\
& \quad + \big(1+B(\tau,0)\big)^{\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)+\frac{s}{2}} \||D|^s u^{ln}(\tau,\cdot)\|_{L^2(\mathbb{R}^n)} \\
& \quad + b(\tau)\big(1+B(\tau,0)\big)^{\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)+1} \|u_t^{ln}(\tau,\cdot)\|_{L^2(\mathbb{R}^n)} \\
& \quad + b(\tau)\big(1+B(\tau,0)\big)^{\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)+1+\frac{s-1}{2}} \||D|^{s-1} u_t^{ln}(\tau,\cdot)\|_{L^2(\mathbb{R}^n)} \Big\}.
\end{aligned}$$

By using the estimates (1.16) to (1.19) of Theorem 1.3 we obtain immediately

$$\|u^{ln}\|_{X(t)} \lesssim \|(u_0,u_1)\|_{\mathcal{A}_{m,s}}. \tag{1.57}$$

We show how to estimate the most complicate norm of the nonlinear part u^{nl} of the solution u which seems to be

$$\||D|^{s-1} u_t^{nl}(t,\cdot)\|_{L^2(\mathbb{R}^n)}.$$

Here we use the estimates (1.27) of Theorem 1.6 for the integral over $\left[0,\frac{t}{2}\right]$ and we take $m=2$ for the integral over $\left[\frac{t}{2},t\right]$. Then we obtain

$$\begin{aligned}
& \||D|^{s-1} u_t^{nl}(t,\cdot)\|_{L^2(\mathbb{R}^n)} \\
& \lesssim \int_0^{\frac{t}{2}} b(t)^{-1} b(\tau)^{-1} (1+B(t,\tau))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-1-\frac{s-1}{2}} \||u(\tau,x)|^p\|_{L^m(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)\cap \dot{H}^{s-1}(\mathbb{R}^n)} d\tau \\
& \quad + \int_{\frac{t}{2}}^t b(t)^{-1} b(\tau)^{-1} (1+B(t,\tau))^{-1-\frac{s-1}{2}} \||u(\tau,x)|^p\|_{L^2(\mathbb{R}^n)\cap \dot{H}^{s-1}(\mathbb{R}^n)} d\tau.
\end{aligned}$$

Now we have to estimate for $0 \leq \tau \leq t$ the following norms:

$$\||u(\tau,x)|^p\|_{L^m(\mathbb{R}^n)}, \quad \||u(\tau,x)|^p\|_{L^2(\mathbb{R}^n)}, \quad \||u(\tau,x)|^p\|_{\dot{H}^{s-1}(\mathbb{R}^n)}.$$

By using the classical Gagliardo-Nirenberg inequality we obtain

$$\begin{aligned}\||u(\tau,x)|^p\|_{L^m(\mathbb{R}^n)} &\lesssim \|u(\tau,\cdot)\|^p_{L^{mp}(\mathbb{R}^n)}\\ &\lesssim \|u(\tau,\cdot)\|^{(1-\theta)p}_{L^2(\mathbb{R}^n)}\,\||D|^s u(\tau,\cdot)\|^{\theta p}_{L^2(\mathbb{R}^n)}.\end{aligned}$$

Here $\theta = \frac{n}{s}\big(\frac{1}{2}-\frac{1}{mp}\big)$ has to fulfil $\theta \in [0,1]$. This is valid for

$$\begin{array}{ll}\frac{2}{m} \leq p & \text{if } \; n \leq 2s,\\ \frac{2}{m} \leq p \leq \frac{2n}{m(n-2s)} & \text{if } \; n > 2s.\end{array}$$

By using the norm of the solution space $X(t)$ we get

$$\|u(\tau,\cdot)\|^p_{L^{mp}(\mathbb{R}^n)} \lesssim (1+B(\tau,0))^{(1-\theta)p\left(-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)\right)+\theta p\left(-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s}{2}\right)}\|u\|^p_{X(t)}.$$

Then

$$\|u(\tau,\cdot)\|^p_{L^{mp}(\mathbb{R}^n)} \lesssim (1+B(\tau,0))^{-\frac{n}{2m}p+\frac{n}{2m}}\|u\|^p_{X(t)}. \tag{1.58}$$

Applying the same ideas leads to

$$\|u(\tau,\cdot)\|^p_{L^{2p}(\mathbb{R}^n)} \lesssim (1+B(\tau,0))^{-\frac{n}{4}p+\frac{n}{4}}\|u\|^p_{X(t)}, \tag{1.59}$$

where

$$\begin{array}{ll}1 \leq p & \text{if } \; n \leq 2s,\\ 1 \leq p \leq \frac{n}{n-2s} & \text{if } \; n > 2s.\end{array}$$

All together implies the following conditions for the exponent p:

$$\begin{array}{ll}\frac{2}{m} \leq p & \text{if } \; n \leq 2s,\\ \frac{2}{m} \leq p \leq \frac{n}{n-2s} & \text{if } \; n > 2s.\end{array}$$

To estimate $\||u(\tau,x)|^p\|_{\dot{H}^{s-1}(\mathbb{R}^n)}$ we use the fractional chain rule from Proposition A.4 and the fractional Gagliardo-Nirenberg inequality. In this way we may conclude

$$\begin{aligned}\||u(\tau,x)|^p\|_{\dot{H}^{s-1}(\mathbb{R}^n)} &= \big\||D|^{s-1}|u(\tau,x)|^p\big\|_{L^2(\mathbb{R}^n)}\\ &\lesssim \big\|u(\tau,\cdot)\big\|^{p-1}_{L^{q_1}(\mathbb{R}^n)}\big\||D|^{s-1}u(\tau,\cdot)\big\|_{L^{q_2}(\mathbb{R}^n)} \text{ for } \; p > \lceil s-1\rceil,\end{aligned}$$

where

$$\frac{p-1}{q_1}+\frac{1}{q_2}=\frac{1}{2}. \tag{1.60}$$

Applying the classical Gagliardo-Nirenberg inequality to $\big\|u(\tau,\cdot)\big\|_{L^{q_1}(\mathbb{R}^n)}$ we get

$$\begin{aligned}\big\|u(\tau,\cdot)\big\|_{L^{q_1}(\mathbb{R}^n)} &\lesssim \big\|u(\tau,\cdot)\big\|^{1-\theta}_{L^2(\mathbb{R}^n)}\big\||D|^s u(\tau,\cdot)\big\|^{\theta}_{L^2(\mathbb{R}^n)}\\ &\lesssim (1+B(\tau,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)(1-\theta)-\left(\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)+\frac{s}{2}\right)\theta}\|u\|_{X(t)}\\ &\lesssim (1+B(\tau,0))^{-\frac{n}{2m}+\frac{n}{2q_1}}\|u\|_{X(t)},\end{aligned}$$

where $\theta = \frac{n}{s}\big(\frac{1}{2} - \frac{1}{q_1}\big) \in [0,1]$. This implies

$$\begin{array}{lll} 2 \leq q_1 & \text{if} & n \leq 2s, \\ 2 \leq q_1 \leq \frac{n}{n-2s} & \text{if} & n > 2s. \end{array}$$

Finally, we arrive at

$$\big\|u(\tau,\cdot)\big\|^{p-1}_{L^{q_1}(\mathbb{R}^n)} \lesssim (1+B(\tau,0))^{\big(-\frac{n}{2m}+\frac{n}{2q_1}\big)(p-1)}\|u\|^{p-1}_{X(t)}. \tag{1.61}$$

We apply the same tools, the fractional Gagliardo-Nirenberg inequality instead, to estimate $\big\||D|^{s-1}u(\tau,\cdot)\big\|_{L^{q_2}(\mathbb{R}^n)}$. So, we obtain

$$\begin{aligned} \big\||D|^{s-1}u(\tau,\cdot)\big\|_{L^{q_2}(\mathbb{R}^n)} &\lesssim \big\|u(\tau,\cdot)\big\|^{1-\tilde{\theta}}_{L^2(\mathbb{R}^n)}\big\||D|^{s}u(\tau,\cdot)\big\|^{\tilde{\theta}}_{L^2(\mathbb{R}^n)} \\ &\lesssim (1+B(\tau,0))^{-\frac{n}{2}\big(\frac{1}{m}-\frac{1}{2}\big)(1-\tilde{\theta})-\big(\frac{n}{2}\big(\frac{1}{m}-\frac{1}{2}\big)+\frac{s}{2}\big)\tilde{\theta}}\|u\|_{X(t)} \\ &\lesssim (1+B(\tau,0))^{-\frac{n}{2m}+\frac{n}{2q_2}-\frac{s-1}{2}}\|u\|_{X(t)}, \end{aligned}$$

where $\tilde{\theta} = \frac{n}{s}\big(\frac{1}{2} - \frac{1}{q_2}\big) + \frac{s-1}{s} \in \big[\frac{s-1}{s},1\big]$. This implies

$$\begin{array}{lll} 2 \leq q_2 & \text{if} & n \leq 2, \\ 2 \leq q_2 \leq \frac{2n}{n-2} & \text{if} & n > 2. \end{array}$$

Consequently, we get

$$\||u(\tau,x)|^p\|_{\dot{H}^{s-1}(\mathbb{R}^n)} \lesssim (1+B(\tau,0))^{-\frac{n}{2m}p+\frac{n}{4}-\frac{s-1}{2}}\|u\|^p_{X(t)}, \tag{1.62}$$

where

$$\begin{array}{lll} 2 \leq q_1, 2 \leq q_2 & \text{if} & n \leq 2, \\ 2 \leq q_1, 2 \leq q_2 \leq \frac{2n}{n-2} & \text{if} & 2 < n \leq 2s, \\ 2 \leq q_1 \leq \frac{n}{n-2s}, 2 \leq q_2 \leq \frac{2n}{n-2} & \text{if} & n > 2s. \end{array}$$

Then we can verify the existence of parameters q_1 and q_2 satisfying the last conditions. For $2 < n \leq 2s$ it is possible to choose $q_2 = \frac{2n}{n-2}$. This choice of q_2 together with the defined relation (1.60) leads to $q_1 = n(p-1) \geq 2$ for $p \geq 1 + \frac{2}{n}$. For $n > 2s$ it is also possible to choose $q_2 = \frac{2n}{n-2}$ which leads to $q_1 = n(p-1)$, too. But, we have the requirement $q_1 \in [2, \frac{2n}{n-2s}]$. This condition on q_1 generates a new upper bound for p, that is,

$$1 + \frac{2}{n} \leq p \leq 1 + \frac{2}{n-2s}. \tag{1.63}$$

This condition is satisfied due to the assumptions in the theorem. Summarizing, the estimates (1.58), (1.59) and (1.62) gives

$$\begin{aligned} &\||D|^{s-1}u^{nl}_t(t,\cdot)\|_{L^2(\mathbb{R}^n)} \\ &\quad\lesssim \|u\|^p_{X(t)}\int_0^{\frac{t}{2}} b(t)^{-1}b(\tau)^{-1}(1+B(t,\tau))^{-\frac{n}{2}\big(\frac{1}{m}-\frac{1}{2}\big)-\frac{s-1}{2}-1}(1+B(\tau,0))^{-\frac{n}{2m}p+\frac{n}{2m}}d\tau \\ &\qquad+\|u\|^p_{X(t)}\int_{\frac{t}{2}}^{t} b(t)^{-1}b(\tau)^{-1}(1+B(t,\tau))^{-\frac{s-1}{2}-1}(1+B(\tau,0))^{-\frac{n}{2m}p+\frac{n}{4}}d\tau. \end{aligned}$$

Now we estimate the last integrals. If $\tau \in \left[0, \frac{t}{2}\right]$, then after using (1.6) we get

$$\begin{aligned}
&\int_0^{\frac{t}{2}} b(t)^{-1}b(\tau)^{-1}(1+B(t,\tau))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s-1}{2}-1}(1+B(\tau,0))^{-\frac{n}{2m}p+\frac{n}{2m}}d\tau \\
&\quad \lesssim b(t)^{-1}(1+B(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s-1}{2}-1}\int_0^{\frac{t}{2}} b(\tau)^{-1}(1+B(\tau,0))^{-\frac{n}{2m}p+\frac{n}{2m}}d\tau \\
&\quad \lesssim b(t)^{-1}(1+B(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s-1}{2}-1},
\end{aligned}$$

where the condition $p > p_{Fuj,m}(n)$ which is satisfied from assumption (1.54) implies $-\frac{n}{2m}p + \frac{n}{2m} < -1$.
If $\tau \in \left[\frac{t}{2}, t\right]$, then after using (1.7) we get

$$\begin{aligned}
&\int_{\frac{t}{2}}^{t} b(t)^{-1}b(\tau)^{-1}(1+B(t,\tau))^{-1-\frac{s-1}{2}}(1+B(\tau,0))^{-\frac{n}{2m}p+\frac{n}{4}}d\tau \\
&\quad \lesssim b(t)^{-1}(1+B(t,0))^{-\frac{n}{2m}p+\frac{n}{4}} \\
&\quad \lesssim b(t)^{-1}(1+B(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s-1}{2}-1}
\end{aligned}$$

for $p > \frac{2m}{n}\left(\frac{s+1}{2}\right)+1$ which is included in condition (1.54) for $s > 3$ and $n \geq 4$. Indeed, from (1.54) we have $p > \lceil s \rceil$. But

$$\lceil s \rceil > s > \Big(\frac{s+1}{2}\Big)+1 > \frac{2m}{n}\Big(\frac{s+1}{2}\Big)+1. \tag{1.64}$$

Hence, we obtain the desired estimate

$$\||D|^{s-1}u_t^{nl}(t,\cdot)\|_{L^2(\mathbb{R}^n)} \lesssim \|u\|_{X(t)}^p b(t)^{-1}(1+B(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s-1}{2}-1}. \tag{1.65}$$

In the same way we can prove

$$\begin{aligned}
\|u^{nl}(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \|u\|_{X(t)}^p(1+B(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)}, && (1.66)\\
\|u_t^{nl}(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \|u\|_{X(t)}^p b(t)^{-1}(1+B(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-1}, && (1.67)\\
\||D|^{s}u^{nl}(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \|u\|_{X(t)}^p(1+B(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s}{2}}. && (1.68)
\end{aligned}$$

All together yields

$$\|u^{nl}\|_{X(t)} \lesssim \|u\|_{X(t)}^p. \tag{1.69}$$

From (1.57) and (1.69) we get (1.55).
To prove (1.56) we recall

$$\|Nu-Nv\|_{X(t)} = \Big\|\int_0^t E_1(t,\tau,x) *_{(x)} \big(|u(\tau,x)|^p - |v(\tau,x)|^p\big)d\tau\Big\|_{X(t)}.$$

We only show how to estimate $\||D|^{s-1}\partial_t(Nu-Nv)(t,\cdot)\|_{L^2(\mathbb{R}^n)}$. In the same way we estimate the other terms appearing in the norm $(|Nu-Nv)\,(t,\cdot)\|_{X(t)}$. It holds

$$\begin{aligned}
&\Big\||D|^{s-1}\partial_t \textstyle\int_0^t E_1(t,\tau,x) *_{(x)} \big(|u(\tau,x)|^p - |v(\tau,x)|^p\big)d\tau\Big\|_{L^2(\mathbb{R}^n)} \\
&\quad \lesssim \textstyle\int_0^{\frac{t}{2}} b(t)^{-1}b(\tau)^{-1}(1+B(t,\tau))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s-1}{2}-1} \\
&\qquad\qquad \times\big\||u(\tau,x)|^p-|v(\tau,x)|^p\big\|_{L^m(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)\cap\dot{H}^{s-1}(\mathbb{R}^n)}d\tau \\
&\quad + \textstyle\int_{\frac{t}{2}}^t b(t)^{-1}b(\tau)^{-1}\big(1+B(t,\tau)\big)^{-\frac{s-1}{2}-1} \\
&\qquad\qquad \times\big\||u(\tau,x)|^p-|v(\tau,x)|^p\big\|_{L^2(\mathbb{R}^n)\cap\dot{H}^{s-1}(\mathbb{R}^n)}\,d\tau.
\end{aligned} \tag{1.70}$$

By Hölder's inequality we conclude for $k = m, 2$ the estimates

$$\big\||u(\tau,x)|^p - |v(\tau,x)|^p\big\|_{L^k(\mathbb{R}^n)} \lesssim \big\|u(\tau,\cdot) - v(\tau,\cdot)\big\|_{L^{kp}(\mathbb{R}^n)}\big(\|u(\tau,\cdot)\|^{p-1}_{L^{kp}(\mathbb{R}^n)} + \|v(\tau,\cdot)\|^{p-1}_{L^{kp}(\mathbb{R}^n)}\big).$$

We apply the classical Gagliardo-Nirenberg inequality to the terms

$$\big\|u(\tau,\cdot) - v(\tau,\cdot)\big\|_{L^{kp}(\mathbb{R}^n)}, \quad \|u(\tau,\cdot)\|_{L^{kp}(\mathbb{R}^n)}, \quad \|v(\tau,\cdot)\|_{L^{kp}(\mathbb{R}^n)}.$$

We get for $0 \leq \tau \leq t$ the estimates

$$\begin{aligned} &\big\||u(\tau,x)|^p - |v(\tau,x)|^p\big\|_{L^2(\mathbb{R}^n)} \\ &\qquad \lesssim (1 + B(\tau,0))^{-\frac{n}{2m}p + \frac{n}{4}}\|u - v\|_{X(t)}\big(\|u\|^{p-1}_{X(t)} + \|v\|^{p-1}_{X(t)}\big), \end{aligned} \tag{1.71}$$

$$\begin{aligned} &\big\||u(\tau,x)|^p - |v(\tau,x)|^p\big\|_{L^m(\mathbb{R}^n)} \\ &\qquad \lesssim (1 + B(\tau,0))^{-\frac{n}{2m}p + \frac{n}{2m}}\|u - v\|_{X(t)}\big(\|u\|^{p-1}_{X(t)} + \|v\|^{p-1}_{X(t)}\big). \end{aligned} \tag{1.72}$$

On the other hand we have after the application of the fractional Leibniz rule in form of Proposition A.2 from the Appendix

$$\begin{aligned} &\big\||u(\tau,x)|^p - |v(\tau,x)|^p\big\|_{\dot{H}^{s-1}(\mathbb{R}^n)} = \big\||D|^{s-1}(|u(\tau,x)|^p - |v(\tau,x)|^p)\big\|_{L^2(\mathbb{R}^n)} \\ &\lesssim \int_0^1 \big\||D|^{s-1}\left\{(u-v)(u - r(u-v))|u - r(u-v)|^{p-2}\right\}\big\|_{L^2(\mathbb{R}^n)} dr \\ &\lesssim \int_0^1 \big\||D|^{s-1}(u-v)\big\|_{L^{q_1}(\mathbb{R}^n)}\big\|(u - r(u-v))|u - r(u-v)|^{p-2}\big\|_{L^{q_2}(\mathbb{R}^n)} dr \\ &\quad + \int_0^1 \big\|u - v\big\|_{L^{q_3}(\mathbb{R}^n)}\big\||D|^{s-1}\left[(u - r(u-v))|u - r(u-v)|^{p-2}\right]\big\|_{L^{q_4}(\mathbb{R}^n)} dr, \end{aligned}$$

where

$$\frac{1}{2} = \frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q_3} + \frac{1}{q_4}. \tag{1.73}$$

By using the fractional Gagliardo-Nirenberg inequality we obtain

$$\begin{aligned} \big\||D|^{s-1}(u-v)\big\|_{L^{q_1}(\mathbb{R}^n)} &\lesssim \big\|(u-v)\big\|^{1-\theta_1}_{L^2(\mathbb{R}^n)}\big\||D|^s(u-v)\big\|^{\theta_1}_{L^2(\mathbb{R}^n)} \\ &\lesssim (1 + B(\tau,0))^{(1-\theta_1)\left(-\frac{n}{2}\left(\frac{1}{m} - \frac{1}{2}\right)\right) + \theta_1\left(-\frac{n}{2}\left(\frac{1}{m} - \frac{1}{2}\right) - \frac{s}{2}\right)}\big\|u - v\big\|_{X(t)} \\ &\lesssim (1 + B(\tau,0))^{-\frac{n}{2m} - \frac{s-1}{2} + \frac{n}{2q_1}}\big\|u - v\big\|_{X(t)}, \end{aligned}$$

where $\theta_1 = \frac{n}{s}\big(\frac{1}{2} - \frac{1}{q_1} + \frac{s-1}{n}\big) \in [\frac{s-1}{s}, 1]$. Then

$$\big\||D|^{s-1}(u-v)\big\|_{L^{q_1}(\mathbb{R}^n)} \lesssim (1 + B(\tau,0))^{-\frac{n}{2m} - \frac{s-1}{2} + \frac{n}{2q_1}}\big\|u - v\big\|_{X(t)}. \tag{1.74}$$

In the same way we use the classical Gagliardo-Nirenberg inequality to the second factor and get

$$\begin{aligned} &\big\|(u - r(u-v))|u - r(u-v)|^{p-2}\big\|_{L^{q_2}(\mathbb{R}^n)} \\ &\qquad \lesssim \big\|u - r(u-v)\big\|^{p-1}_{L^{q_2(p-1)}(\mathbb{R}^n)} \\ &\qquad \lesssim \big\|u - r(u-v)\big\|^{(1-\theta_2)(p-1)}_{L^2(\mathbb{R}^n)}\big\||D|^s\left(u - r(u-v)\right)\big\|^{\theta_2(p-1)}_{L^2(\mathbb{R}^n)} \\ &\qquad \lesssim (1 + B(\tau,0))^{(1-\theta_2)(p-1)\left(-\frac{n}{2}\left(\frac{1}{m} - \frac{1}{2}\right)\right) + \theta_2(p-1)\left(-\frac{n}{2}\left(\frac{1}{m} - \frac{1}{2}\right) - \frac{s}{2}\right)}\big\|u - v\big\|_{X(t)} \\ &\qquad \lesssim (1 + B(\tau,0))^{-\frac{n}{2m}(p-1) + \frac{n}{2q_2}}\big\|u - r(u-v)\big\|^{p-1}_{X(t)}, \end{aligned}$$

where $\theta_2 = \frac{n}{s}\big(\frac{1}{2} - \frac{1}{q_2(p-1)}\big) \in [0,1]$. So,

$$\big\|u - r(u-v)\big\|^{p-1}_{L^{q_2}(\mathbb{R}^n)} \lesssim (1+B(\tau,0))^{-\frac{n}{2m}(p-1)+\frac{n}{2q_2}}\big\|u - r(u-v)\big\|^{p-1}_{X(t)}. \tag{1.75}$$

Now we discuss the choice of parameters θ_1 and θ_2. From the definition we get the following conditions:

$$\frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{2}$$

and

$$\begin{array}{llll} \frac{1}{q_1} \leq \frac{1}{2} & \text{and} & \frac{1}{q_2} \leq \frac{p-1}{2} & \text{if} \quad n \leq 2, \\ \frac{n-2}{2n} \leq \frac{1}{q_1} \leq \frac{1}{2} & \text{and} & \frac{1}{q_2} \leq \frac{p-1}{2} & \text{if} \quad 2 < n \leq 2s, \\ \frac{n-2}{2n} \leq \frac{1}{q_1} \leq \frac{1}{2} & \text{and} & \frac{n-2s}{2n}(p-1) \leq \frac{1}{q_2} \leq \frac{p-1}{2} & \text{if} \quad n > 2s. \end{array}$$

We distinguish between two cases.
If $2 < n \leq 2s$, then we can choose $q_1 = \frac{2n}{n-2}$. From q_1 we can fix $q_2 = n$ which fulfils also $\frac{1}{q_2} \leq \frac{p-1}{2}$ for $p \geq 1 + \frac{2}{n}$.
If $n > 2s$, then we can choose $q_1 = \frac{2n}{n-2}$ which fulfils $\frac{n-2}{2n} \leq \frac{1}{q_1} \leq \frac{1}{2}$. Then we fix $q_2 = n$ which also fulfils $\frac{n-2s}{2n}(p-1) \leq \frac{1}{q_2} \leq \frac{p-1}{2}$ for $p \leq 1 + \frac{2}{n-2s}$. Then after getting (1.74) and (1.75) we may conclude

$$\begin{aligned} &\big\||D|^{s-1}(u-v)\big\|_{L^{q_1}(\mathbb{R}^n)}\big\|(u-r(u-v))|u-r(u-v)|^{p-2}\big\|_{L^{q_2}(\mathbb{R}^n)} \\ &\qquad \lesssim (1+B(\tau,0))^{-\frac{n}{2m}p+\frac{n}{4}-\frac{s-1}{2}}\big\|u-v\big\|_{X(t)}\big\|u-r(u-v)\big\|^{p-1}_{X(t)}. \end{aligned} \tag{1.76}$$

Now we turn to estimate the second integral. We have for the first term of the second integral

$$\begin{aligned} \big\|u-v\big\|_{L^{q_3}(\mathbb{R}^n)} &\lesssim \big\|(u-v)\big\|^{1-\theta_3}_{L^2(\mathbb{R}^n)}\big\||D|^s(u-v)\big\|^{\theta_3}_{L^2(\mathbb{R}^n)} \\ &\lesssim (1+B(\tau,0))^{(1-\theta_3)\left(-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)\right)+\theta_3\left(-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s}{2}\right)}\big\|u-v\big\|_{X(t)} \\ &\lesssim (1+B(\tau,0))^{-\frac{n}{2m}+\frac{n}{2q_3}}\big\|u-v\big\|_{X(t)}. \end{aligned}$$

Here $\theta_3 = \frac{n}{s}\big(\frac{1}{2} - \frac{1}{q_3}\big) \in [0,1]$. So,

$$\big\|u-v\big\|_{L^{q_3}(\mathbb{R}^n)} \lesssim (1+B(\tau,0))^{-\frac{n}{2m}+\frac{n}{2q_3}}\big\|u-v\big\|_{X(t)}. \tag{1.77}$$

Now we estimate the second term. After using Proposition A.4 for $p-1 > \lceil s-1 \rceil$ we arrive at the estimate

$$\begin{aligned} &\big\||D|^{s-1}\left[(u-r(u-v))|u-r(u-v)|^{p-2}\right]\big\|_{L^{q_4}(\mathbb{R}^n)} \\ &\qquad \lesssim \|u-r(u-v)\|^{p-2}_{L^{q_5}(\mathbb{R}^n)}\||D|^{s-1}(u-r(u-v))\|_{L^{q_6}(\mathbb{R}^n)}, \end{aligned}$$

where $\frac{1}{q_4} = \frac{p-2}{q_5} + \frac{1}{q_6}$.
Then, after using the classical Gagliardo-Nirenberg inequality we get for the first term the following estimate:

$$\begin{aligned} &\|u-r(u-v)\|^{p-2}_{L^{q_5}(\mathbb{R}^n)} \lesssim \|u-r(u-v)\|^{(p-2)(1-\theta_5)}_{L^2(\mathbb{R}^n)}\||D|^s(u-r(u-v))\|^{(p-2)\theta_5}_{L^2(\mathbb{R}^n)} \\ &\qquad \lesssim (1+B(\tau,0))^{\left(-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)\right)(p-2)(1-\theta_5)+\left(-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s}{2}\right)(p-2)\theta_5}\big\|u-r(u-v)\big\|^{p-2}_{X(t)} \\ &\qquad \lesssim (1+B(\tau,0))^{-\frac{n}{2m}(p-2)+\frac{n}{2q_5}(p-2)}\big\|u-r(u-v)\big\|^{p-2}_{X(t)}, \end{aligned}$$

where $\theta_5 = \frac{n}{s}\big(\frac{1}{2} - \frac{1}{q_5}\big) \in [0,1]$. Applying the fractional Gagliardo-Nirenberg inequality from Proposition A.1 we may conclude for the second term of the last right-hand side the estimate

$$\begin{aligned}
\||D|^{s-1}(u-r(u-v))\|_{L^{q_6}(\mathbb{R}^n)} &\lesssim \|u-r(u-v)\|^{1-\theta_6}_{L^2(\mathbb{R}^n)}\||D|^s(u-r(u-v))\|^{\theta_6}_{L^2(\mathbb{R}^n)}\\
&\lesssim (1+B(\tau,0))^{\left(-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)\right)(1-\theta_6)+\left(-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s}{2}\right)\theta_6}\big\|u-r(u-v)\big\|_{X(t)}\\
&\lesssim (1+B(\tau,0))^{-\frac{n}{2m}+\frac{n}{2q_6}-\frac{s-1}{2}}\big\|u-r(u-v)\big\|_{X(t)},
\end{aligned}$$

where $\theta_6 = \frac{n}{s}\big(\frac{1}{2} - \frac{1}{q_6}\big) + \frac{s-1}{s} \in [\frac{s-1}{s},1]$.
Summarizing gives

$$\begin{aligned}
&\Big\||D|^{s-1}\left[(u-r(u-v))|u-r(u-v)|^{p-2}\right]\Big\|_{L^{q_4}(\mathbb{R}^n)}\\
&\quad\lesssim (1+B(\tau,0))^{-\frac{n}{2m}(p-2)+\frac{n}{2q_5}(p-2)-\frac{n}{2m}+\frac{n}{2q_6}-\frac{s-1}{2}}\big\|u-r(u-v)\big\|^{p-1}_{X(t)}\\
&\quad= (1+B(\tau,0))^{-\frac{n}{2m}(p-1)+\frac{n}{2q_4}-\frac{s-1}{2}}\big\|u-r(u-v)\big\|^{p-1}_{X(t)}.
\end{aligned}$$

Now we have to verify if we can choose in a suitable way all parameters q_3 to q_6. The conditions

$$\frac{1}{q_3}+\frac{1}{q_4}=\frac{1}{2} \text{ and } \frac{p-2}{q_5}+\frac{1}{q_6}=\frac{1}{q_4} \text{ imply } \frac{1}{q_3}+\frac{p-2}{q_5}+\frac{1}{q_6}=\frac{1}{2}.$$

Moreover, we get the conditions

$$\begin{array}{lll}
\frac{1}{q_3}\le\frac{1}{2},\ \frac{1}{q_5}\le\frac{1}{2} & \text{and } \frac{n-2}{2n}\le\frac{1}{q_6}\le\frac{1}{2} & \text{if } 2<n\le 2s,\\
\frac{n-2s}{2n}\le\frac{1}{q_3}\le\frac{1}{2},\ \frac{n-2s}{2n}\le\frac{1}{q_5}\le\frac{1}{2} & \text{and } \frac{n-2}{2n}\le\frac{1}{q_6}\le\frac{1}{2} & \text{if } \quad n>2s.
\end{array}$$

One possibility to choose the parameters q_3, q_4, q_5 and q_6 satisfying the last conditions is

$$q_3 = n(p-1),\ \ q_4=\frac{2n(p-1)}{n(p-1)-2},\ \ q_5=n(p-1),\ \ q_6=\frac{2n}{n-2}.$$

This choice implies the condition $1+\frac{2}{n}\le p\le 1+\frac{2}{n-2s}$ for $n>2$ which is contained in (1.54).
Again using all these estimates implies

$$\begin{aligned}
&\big\|u-v\big\|_{L^{q_3}(\mathbb{R}^n)}\Big\||D|^{s-1}\left[(u-r(u-v))|u-r(u-v)|^{p-2}\right]\Big\|_{L^{q_4}(\mathbb{R}^n)}\\
&\quad\lesssim (1+B(\tau,0))^{-\frac{n}{2m}p+\frac{n}{4}-\frac{s-1}{2}}\big\|u-v\big\|_{X(t)}\big\|u-r(u-v)\big\|^{p-1}_{X(t)}.
\end{aligned}$$

Consequently, we obtain for $0\le\tau\le t$ the estimate

$$\begin{aligned}
&\big\||u(\tau,x)|^p-|v(\tau,x)|^p\big\|_{\dot H^{s-1}(\mathbb{R}^n)}\\
&\quad\lesssim \int_0^1 (1+B(\tau,0))^{-\frac{n}{2m}p+\frac{n}{4}-\frac{s-1}{2}}\big\|u-v\big\|_{X(t)}\big\|u-r(u-v)\big\|^{p-1}_{X(t)}dr\\
&\quad\lesssim \int_0^1 (1+B(\tau,0))^{-\frac{n}{2m}p+\frac{n}{4}-\frac{s-1}{2}}\big\|u-v\big\|_{X(t)}\big(\|u\|^{p-1}_{X(t)}+\|v\|^{p-1}_{X(t)}\big)dr\\
&\quad\lesssim (1+B(\tau,0))^{-\frac{n}{2m}p+\frac{n}{4}-\frac{s-1}{2}}\big\|u-v\big\|_{X(t)}\big(\|u\|^{p-1}_{X(t)}+\|v\|^{p-1}_{X(t)}\big).
\end{aligned}$$

Finally, we get for $0 \le \tau \le t$ the following estimate:

$$\begin{aligned} \big\||u(\tau,x)|^p-|v(\tau,x)|^p\big\|_{\dot H^{s-1}(\mathbb{R}^n)} &\lesssim (1+B(\tau,0))^{-\frac{n}{2m}p+\frac{n}{4}-\frac{s-1}{2}} \\ &\qquad \times\big\|u-v\big\|_{X(t)}\big(\|u\|^{p-1}_{X(t)}+\|v\|^{p-1}_{X(t)}\big). \end{aligned} \tag{1.78}$$

Taking account of (1.70) we have

$$\begin{aligned} &\Big\||D|^s\partial_t\int_0^t E_1(t,\tau,x)*_{(x)}\big(|u(\tau,x)|^p-|v(\tau,x)|^p\big)d\tau\Big\|_{L^2(\mathbb{R}^n)} \\ &\lesssim \int_0^{\frac{t}{2}} b(t)^{-1}b(\tau)^{-1}(1+B(t,\tau))^{-\frac{n}{2}\left(\frac{1}{m}-\frac12\right)-\frac{s-1}{2}-1}(1+B(\tau,0))^{-\frac{n}{2m}p+\frac{n}{2m}} \\ &\qquad\qquad \times\|u-v\|_{X(t)}\big(\|u\|^{p-1}_{X(t)}+\|v\|^{p-1}_{X(t)}\big)d\tau \\ &\quad+\int_{\frac{t}{2}}^t b(t)^{-1}b(\tau)^{-1}(1+B(t,\tau))^{-\frac{s-1}{2}-1}(1+B(\tau,0))^{-\frac{n}{2m}p+\frac{n}{4}} \\ &\qquad\qquad \times\big\|u-v\big\|_{X(t)}\big(\|u\|^{p-1}_{X(t)}+\|v\|^{p-1}_{X(t)}\big)d\tau. \end{aligned}$$

Following the same steps to estimate the term $\||D|^{s-1}u_t^{nl}(t,\cdot)\|_{L^2(\mathbb{R}^n)}$ we can immediately conclude

$$\begin{aligned} &\||D|^{s-1}\partial_t(Nu-Nv)(t,\cdot)\|_{L^2(\mathbb{R}^n)} \\ &\quad\lesssim (1+B(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac12\right)-\frac{s-1}{2}-1}\big\|u-v\big\|_{X(t)}\big(\|u\|^{p-1}_{X(t)}+\|v\|^{p-1}_{X(t)}\big). \end{aligned}$$

In the same way we can derive the estimates

$$\begin{aligned} \||D|^s(Nu-Nv)(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim (1+B(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac12\right)-\frac{s}{2}} \\ &\qquad\times\big\|u-v\big\|_{X(t)}\big(\|u\|^{p-1}_{X(t)}+\|v\|^{p-1}_{X(t)}\big), \\ \|\partial_t(Nu-Nv)(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim b(t)^{-1}(1+B(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac12\right)-1} \\ &\qquad\times\big\|u-v\big\|_{X(t)}\big(\|u\|^{p-1}_{X(t)}+\|v\|^{p-1}_{X(t)}\big), \\ \|(Nu-Nv)(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim (1+B(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac12\right)}\big\|u-v\big\|_{X(t)}\big(\|u\|^{p-1}_{X(t)}+\|v\|^{p-1}_{X(t)}\big). \end{aligned}$$

The proof is completed if we replace all these estimates into the definition of the norm of solution space $X(t)$ to get (1.56). □

If $s\in(1,3)$ or $n=3$, then the condition $p>\frac{2m}{n}\left(\frac{s+1}{2}\right)+1$ is not included in (1.54). For this reason we will additionally suppose $p>\max\{\frac{2m}{n}\left(\frac{s+1}{2}\right)+1;\lceil s\rceil\}$. We introduce the following lemma to weaken the last condition.

Lemma 1.2. *The following estimates hold for the solutions of (1.20):*

$$\big\||D|^{s-1}v_t(t,\cdot)\big\|_{L^2(\mathbb{R}^n)}\lesssim b(t)^{-1}b(\tau)^{-1}B(t,\tau)^{-1+\frac{\varepsilon}{2}}\|g(\tau,\cdot)\|_{\dot H^{s-1+\varepsilon}(\mathbb{R}^n)\cap\dot H^{s-1}(\mathbb{R}^n)}, \tag{1.79}$$

$$\big\||D|^s v(t,\cdot)\big\|_{L^2(\mathbb{R}^n)} \lesssim b(\tau)^{-1}B(t,\tau)^{-\frac12}\|g(\tau,\cdot)\|_{\dot H^{s-1}(\mathbb{R}^n)}, \tag{1.80}$$

where $s>1$ and ε is a small, positive and real number.

Proof. The proof is based on the last section of the paper [4], exactly on the Lemmas 7.2 and 7.3. Indeed, for showing the estimate (1.79) we split the extended phase space into two zones, those for large and for small frequencies. For large frequencies $|\xi| \geq \Theta = \Theta(t,\tau)$ we have

$$\big\||\xi|^{s-1}\partial_t\hat{\Phi}(t,\tau,\cdot)\hat{g}(\tau,\cdot)\big\|_{L^2_{\{|\xi|\geq\Theta\}}(\mathbb{R}^n)} \leq \big\|\partial_t\hat{\Phi}(t,\tau,\cdot)\big\|_{L^\infty_{\{|\xi|\geq\Theta\}}(\mathbb{R}^n)}\big\||\xi|^{s-1}\hat{g}(\tau,\cdot)\big\|_{L^2_{\{|\xi|\geq\Theta\}}(\mathbb{R}^n)}.$$

From Lemma 7.2 in [4] one can get immediately

$$\big\||\xi|^{s-1}\partial_t\hat{\Phi}(t,\tau,\cdot)\hat{g}(\tau,\cdot)\big\|_{L^2_{\{|\xi|\geq\Theta\}}(\mathbb{R}^n)} \lesssim b(\tau)^{-1}\Big(\frac{\lambda(\tau)}{\lambda(t)}\Big)^{1-2\delta}\|g(\tau,\cdot)\|_{\dot{H}^{s-1}(\mathbb{R}^n)}. \tag{1.81}$$

For small frequencies $|\xi| \leq \Theta = \Theta(t,\tau)$ we have

$$\begin{aligned}&\big\||\xi|^{s-1}\partial_t\hat{\Phi}(t,\tau,\cdot)\hat{g}(\tau,\cdot)\big\|_{L^2_{\{|\xi|\leq\Theta\}}(\mathbb{R}^n)}\\ &\qquad\leq \big\||\xi|^{-\varepsilon}\partial_t\hat{\Phi}(t,\tau,\cdot)\big\|_{L^\infty_{\{|\xi|\leq\Theta\}}(\mathbb{R}^n)}\big\||\xi|^{s-1+\varepsilon}\hat{g}(\tau,\cdot)\big\|_{L^2_{\{|\xi|\leq\Theta\}}(\mathbb{R}^n)}.\end{aligned}$$

From the estimate (7.29) in [4] we obtain

$$\begin{aligned}||\xi|^{-\varepsilon}\partial_t\hat{\Phi}(t,\tau,\xi)| &\lesssim b(t)^{-1}b(\tau)^{-1}|\xi|^{2-\varepsilon}\exp(-C|\xi|^2B(t,\tau))\\ &\lesssim b(t)^{-1}b(\tau)^{-1}B(t,\tau)^{-1+\frac{\varepsilon}{2}}.\end{aligned}$$

Then

$$\begin{aligned}&\big\||\xi|^{s-1}\partial_t\hat{\Phi}(t,\tau,\cdot)\hat{g}(\tau,\cdot)\big\|_{L^2_{\{|\xi|\Theta\}}(\mathbb{R}^n)}\\ &\qquad\lesssim\leq b(t)^{-1}b(\tau)^{-1}B(t,\tau)^{-1+\frac{\varepsilon}{2}}\big\||\xi|^{s-1+\varepsilon}\hat{g}(\tau,\cdot)\big\|_{L^2_{\{|\xi|\leq\Theta\}}(\mathbb{R}^n)}.\end{aligned} \tag{1.82}$$

Consequently, (1.81) and (1.82) leads to (1.80).
Analogously, for proving the second estimate (1.80) we have for large frequencies

$$\big\||\xi|^{s}\hat{\Phi}(t,\tau,\cdot)\hat{g}(\tau,\cdot)\big\|_{L^2_{\{|\xi|\geq\Theta\}}(\mathbb{R}^n)} \lesssim b(\tau)^{-1}\Big(\frac{\lambda(\tau)}{\lambda(t)}\Big)^{1-2\delta}\|g(\tau,\cdot)\|_{\dot{H}^{s-1}(\mathbb{R}^n)}. \tag{1.83}$$

Similar to the proof of (1.82) using the estimate (7.29) from [4] one can conclude

$$\big\||\xi|^{s}\hat{\Phi}(t,\tau,\cdot)\hat{g}(\tau,\cdot)\big\|_{L^2_{\{|\xi|\leq\Theta\}}(\mathbb{R}^n)} \leq \big\||\xi|\hat{\Phi}(t,\tau,\cdot)\big\|_{L^\infty_{\{|\xi|\leq\Theta\}}(\mathbb{R}^n)}\big\||\xi|^{s-1}\hat{g}(\tau,\cdot)\big\|_{L^2_{\{|\xi|\leq\Theta\}}(\mathbb{R}^n)},$$

and

$$\begin{aligned}||\xi|\partial_t\hat{\Phi}(t,\tau,\xi)| &\lesssim b(\tau)^{-1}|\xi|\exp(-C|\xi|^2B(t,\tau))\\ &\lesssim b(\tau)^{-1}B(t,\tau)^{-\frac{1}{2}}.\end{aligned}$$

Then

$$\big\||\xi|^{s}\hat{\Phi}(t,\tau,\cdot)\hat{g}(\tau,\cdot)\big\|_{L^2_{\{|\xi|\leq\Theta\}}(\mathbb{R}^n)} \lesssim b(\tau)^{-1}B(t,\tau)^{-\frac{1}{2}}\big\||\xi|^{s-1}\hat{g}(\tau,\cdot)\big\|_{L^2_{\{|\xi|\leq\Theta\}}(\mathbb{R}^n)}. \tag{1.84}$$

Then (1.83) and (1.84) leads to (1.80). □

Theorem 1.11. *Let $n \geq 3$ and $s \in (1,3)$ be a non-integer number. The data are supposed to satisfy*

$$(u_0,u_1) \in \mathcal{A}_{m,s,\varepsilon} = \big(H^s(\mathbb{R}^n) \cap L^m(\mathbb{R}^n)\big) \times \big(H^{s-1+\varepsilon}(\mathbb{R}^n) \cap L^m(\mathbb{R}^n)\big),$$

where $m \in [1,2)$. Finally, the following conditions are satisfied for the exponent p:

$$\begin{aligned} &\max\left\{\tfrac{2m}{n}+1;\lceil s\rceil,2\right\} < \; p && \text{if} \quad s \in [\tfrac{n}{2},\tfrac{n}{2}+1],\\ &\max\left\{\tfrac{2m}{n}+1;\lceil s\rceil,2\right\} < \; p < 1+\tfrac{2}{n-2s} && \text{if} \quad s \in (1,\tfrac{n}{2}). \end{aligned} \tag{1.85}$$

Then, there exists a small constant ϵ_0 such that if

$$\|(u_0,u_1)\|_{\mathcal{A}_{m,s,\varepsilon}} \leq \epsilon_0,$$

then there exists a uniquely determined globally (in time) energy solution to (1.53) in

$$\mathcal{C}\big([0,\infty),H^s(\mathbb{R}^n)\big) \cap \mathcal{C}^1\big([0,\infty),H^{s-1+\varepsilon}(\mathbb{R}^n)\big).$$

Furthermore, the solution satisfies the estimates:

$$\begin{aligned} \|u(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \big(1+B(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)}\|(u_0,u_1)\|_{\mathcal{A}_{m,s,\varepsilon}},\\ \|u_t(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim b(t)^{-1}\big(1+B(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-1}\|(u_0,u_1)\|_{\mathcal{A}_{m,s,\varepsilon}},\\ \||D|^s u(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \big(1+B(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s}{2}}\|(u_0,u_1)\|_{\mathcal{A}_{m,s,\varepsilon}},\\ \||D|^{s-1} u_t(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim b(t)^{-1}\big(1+B(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-1-\frac{s-1}{2}}\|(u_0,u_1)\|_{\mathcal{A}_{m,s,\varepsilon}}. \end{aligned}$$

Proof. Using Lemma 1.2 we modify the estimate of $\||D|^{s-1}u_t^{nl}(t,\cdot)\|_{L^2(\mathbb{R}^n)}$ in the proof of Theorem 1.10 in the integral over $[\frac{t}{2},t]$ as follows:

$$\begin{aligned} &\||D|^{s-1}u_t^{nl}(t,\cdot)\|_{L^2(\mathbb{R}^n)}\\ &\lesssim \int_0^{\frac{t}{2}} b(t)^{-1}b(\tau)^{-1}(1+B(t,\tau))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-1-\frac{s-1}{2}}\||u(\tau,x)|^p\|_{L^m(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)\cap \dot{H}^{s-1}(\mathbb{R}^n)}d\tau\\ &\qquad + \int_{\frac{t}{2}}^t b(t)^{-1}b(\tau)^{-1}B(t,\tau)^{-1+\frac{\varepsilon}{2}}\||u(\tau,x)|^p\|_{\dot{H}^{s-1}(\mathbb{R}^n)\cap \dot{H}^{s-1+\varepsilon}(\mathbb{R}^n)}d\tau. \end{aligned}$$

Using the fractional chain rule for $0 \leq \tau \leq t$ we get

$$\||u(\tau,x)|^p\|_{\dot{H}^{s-1}(\mathbb{R}^n)} \lesssim (1+B(\tau,0))^{-\frac{n}{2m}p+\frac{n}{4}-\frac{s-1}{2}}\|u\|^p_{X(t)}, \tag{1.86}$$

$$\||u(\tau,x)|^p\|_{\dot{H}^{s-1+\varepsilon}(\mathbb{R}^n)} \lesssim (1+B(\tau,0))^{-\frac{n}{2m}p+\frac{n}{4}-\frac{s-1+\varepsilon}{2}}\|u\|^p_{X(t)}, \tag{1.87}$$

where

$$\lceil s-1+\varepsilon\rceil < p \qquad \text{and} \qquad \frac{n}{2}+1 \leq p < 1+\frac{2}{n-2s}.$$

Consequently, we obtain for $\tau \in [0,\frac{t}{2}]$ the estimate

$$\begin{aligned} &\int_0^{\frac{t}{2}} b(t)^{-1}b(\tau)^{-1}(1+B(t,\tau))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-1-\frac{s-1}{2}}\||u(\tau,x)|^p\|_{L^m(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)\cap \dot{H}^{s-1}(\mathbb{R}^n)}d\tau\\ &\qquad \lesssim \|u\|^p_{X(t)}b(t)^{-1}(1+B(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s-1}{2}-1}, \end{aligned}$$

where $p > p_{Fuj,m}(n)$ is used.
Now, for the second part of the integral over the interval $[\frac{t}{2}, t]$, we have

$$\begin{aligned}
&\int_{\frac{t}{2}}^{t} b(t)^{-1}b(\tau)^{-1}(1+B(t,\tau))^{-1-\frac{\varepsilon}{2}}(1+B(\tau,0))^{-\frac{n}{2m}p+\frac{n}{4}-\frac{s-1}{2}}d\tau \\
&\quad\lesssim b(t)^{-1}(1+B(t,0))^{-\frac{n}{2m}p+\frac{n}{4}-\frac{s-1}{2}+\frac{\varepsilon}{2}} \\
&\quad\lesssim b(t)^{-1}(1+B(t,0))^{-\frac{n}{2}(\frac{1}{m}-\frac{1}{2})-\frac{s-1}{2}-1},
\end{aligned}$$

where $p > \frac{2m}{n}\left(1+\frac{\varepsilon}{2}\right)+1$ for a sufficiently small positive ε. This condition is included in (1.85). Finally, we get

$$\||D|^{s-1}u_t^{nl}(t,\cdot)\|_{L^2(\mathbb{R}^n)} \lesssim \|u\|_{X(t)}^p b(t)^{-1}(1+B(t,0))^{-\frac{n}{2}(\frac{1}{m}-\frac{1}{2})-\frac{s-1}{2}-1}. \tag{1.88}$$

The estimate (1.88) with (1.66) to (1.68) complete the proof of the first inequality (1.55). Analogously, we can use Lemma 1.2 to prove the following estimate:

$$\begin{aligned}
\||D|^{s-1}\partial_t(Nu-Nv)(t,\cdot)\|_{L^2(\mathbb{R}^n)} \lesssim\ & b(t)^{-1}(1+B(t,0))^{-\frac{n}{2}(\frac{1}{m}-\frac{1}{2})-\frac{s-1}{2}-1}\ {}_{X(t)} \\
&\times\big\|u-v\big\|\big(\big\|u\big\|_{X(t)}^{p-1}+\big\|v\big\|_{X(t)}^{p-1}\big)
\end{aligned}$$

for $p > \max\{\lceil s+\varepsilon\rceil; 2\}$. We combine the last estimate with the proof of (1.56). In this way the proof is completed. $\square$

1.5. Large regular data

We turn again to the Cauchy problem

$$u_{tt} - \Delta u + b(t)u_t = |u|^p, \;\; u(0,x) = u_0(x), \;\; u_t(0,x) = u_1(x). \tag{1.89}$$

This case has been classified to benefit from the embedding in $L^\infty(\mathbb{R}^n)$, where the data are supposed to have a high regularity, this means, that

$$(u_0,u_1) \in \mathcal{A}_{m,s} := \big(H^s(\mathbb{R}^n)\cap L^m(\mathbb{R}^n)\big)\times\big(H^{s-1}(\mathbb{R}^n)\cap L^m(\mathbb{R}^n)\big), \;\; s > \frac{n}{2}+1.$$

Having different tools in hand to prove the global (in time) existence of small data solutions leads to different results or conditions on the power nonlinearity.

Theorem 1.12. *Let us assume $n \geq 4$. The data satisfy the condition $(u_0,u_1) \in \mathcal{A}_{m,s}$ with $s > \frac{n}{2}+1$. Finally, the exponent p satisfies the condition*

$$p > \max\Big\{s; \frac{2m}{n}\Big(\frac{2}{2-m}\Big)+1\Big\}. \tag{1.90}$$

Then, there exists a small positive constant ϵ_0 such that if

$$\|(u_0,u_1)\|_{\mathcal{A}_{m,s}} \leq \epsilon_0,$$

then there exists a uniquely determined globally (in time) energy solution to (1.89) in

$$\mathcal{C}\big([0,\infty),H^s(\mathbb{R}^n)\big)\cap\mathcal{C}^1\big([0,\infty),H^{s-1}(\mathbb{R}^n)\big).$$

The solution satisfies the following estimates:

$$\begin{aligned}
\|u(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \big(1+B(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)}\|(u_0,u_1)\|_{\mathcal{A}_{m,s}},\\
\|u_t(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim b(t)^{-1}\big(1+B(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-1}\|(u_0,u_1)\|_{\mathcal{A}_{m,s}},\\
\||D|^s u(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \big(1+B(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s}{2}}\|(u_0,u_1)\|_{\mathcal{A}_{m,s}},\\
\||D|^{s-1} u_t(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim b(t)^{-1}\big(1+B(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s-1}{2}-1}\|(u_0,u_1)\|_{\mathcal{A}_{m,s}}.
\end{aligned}$$

Proof. To prove the theorem we follow the same approach used in the proofs of the previous theorems. Firstly, we define the solution space and its norm as follows:

$$X(t)=\mathcal{C}([0,t],H^s(\mathbb{R}^n))\cap\mathcal{C}^1([0,t],H^{s-1}(\mathbb{R}^n))$$

with the norm

$$\begin{aligned}
\|u\|_{X(t)}=\sup_{\tau\in[0,t]}\Big\{&\big(1+B(\tau,0)\big)^{\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)}\|u(\tau,\cdot)\|_{L^2(\mathbb{R}^n)}\\
&+b(\tau)\big(1+B(\tau,0)\big)^{\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)+1}\|u_t(\tau,\cdot)\|_{L^2(\mathbb{R}^n)}\\
&+\big(1+B(\tau,0)\big)^{\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)+\frac{s}{2}}\||D|^s u(\tau,\cdot)\|_{L^2(\mathbb{R}^n)}\\
&+b(\tau)\big(1+B(\tau,0)\big)^{\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)+1+\frac{s-1}{2}}\||D|^{s-1} u_t(\tau,\cdot)\|_{L^2(\mathbb{R}^n)}\Big\}.
\end{aligned}$$

We shall prove the following inequalities:

$$\|Nu\|_{X(t)}\lesssim\|(u_0,u_1)\|_{\mathcal{A}_{m,s}}+\|u\|^p_{X(t)}, \tag{1.91}$$

$$\|Nu-Nv\|_{X(t)}\lesssim\|u-v\|_{X(t)}\big(\|u\|^{p-1}_{X(t)}+\|v\|^{p-1}_{X(t)}\big), \tag{1.92}$$

where the operator N is defined as in (1.30). We may conclude immediately

$$\|u^{ln}\|_{X(t)}\lesssim\|(u_0,u_1)\|_{\mathcal{A}_{m,s}}. \tag{1.93}$$

To treat the nonlinear part we need to estimate

$$\||u(\tau,x)|^p\|_{L^2(\mathbb{R}^n)},\ \ \||u(\tau,x)|^p\|_{L^m(\mathbb{R}^n)},\ \ \||u(\tau,x)|^p\|_{\dot{H}^{s-1}(\mathbb{R}^n)}.$$

By using the Gagliardo-Nirenberg inequality from Corollary A.1 one can get for $p\geq\frac{2}{m}$ and $0\leq\tau\leq t$ the estimates

$$\|u(\tau,\cdot)\|^p_{L^{mp}(\mathbb{R}^n)}\lesssim(1+B(\tau,0))^{-\frac{n}{2m}p+\frac{n}{2m}}\|u\|^p_{X(t)}, \tag{1.94}$$

and

$$\|u(\tau,\cdot)\|^p_{L^{2p}(\mathbb{R}^n)}\lesssim(1+B(\tau,0))^{-\frac{n}{2m}p+\frac{n}{4}}\|u\|^p_{X(t)}. \tag{1.95}$$

To estimate $\||u(\tau,x)|^p\|_{\dot H^{s-1}(\mathbb{R}^n)}$ we use the rules for estimating fractional powers for $s-1>\frac{n}{2}$. More details can be found in Corollary A.3. Then we derive the following estimates:

$$\begin{aligned}\||u(\tau,x)|^p\|_{\dot H^{s-1}(\mathbb{R}^n)} &\lesssim \|u(\tau,\cdot)\|_{\dot H^{s-1}(\mathbb{R}^n)}\|u(\tau,\cdot)\|^{p-1}_{L^\infty(\mathbb{R}^n)}\\ &\lesssim \|u(\tau,\cdot)\|_{\dot H^{s-1}(\mathbb{R}^n)}\|u(\tau,\cdot)\|^{p-1}_{H^{s-1}(\mathbb{R}^n)}\\ &\lesssim \|u(\tau,\cdot)\|_{\dot H^{s-1}(\mathbb{R}^n)}\big(\|u(\tau,\cdot)\|_{L^2(\mathbb{R}^n)}+\|u(\tau,\cdot)\|_{\dot H^{s-1}(\mathbb{R}^n)}\big)^{p-1}\\ &\lesssim \|u(\tau,\cdot)\|^p_{\dot H^{s-1}(\mathbb{R}^n)}+\|u(\tau,\cdot)\|_{\dot H^{s-1}(\mathbb{R}^n)}\|u(\tau,\cdot)\|^{p-1}_{L^2(\mathbb{R}^n)}.\end{aligned}$$

Taking into consideration the definition of the norm of the solution space $X(t)$ we obtain for $0\le\tau\le t$ the following estimates:

$$\|u(\tau,\cdot)\|^p_{\dot H^{s-1}(\mathbb{R}^n)}\lesssim(1+B(\tau,0))^{\left(-\frac{n}{2}\left(\frac{1}{m}-\frac12\right)-\frac{s-1}{2}\right)p}\|u\|^p_{X(t)},$$

$$\|u(\tau,\cdot)\|_{\dot H^{s-1}(\mathbb{R}^n)}\|u(\tau,\cdot)\|^{p-1}_{L^2(\mathbb{R}^n)}\lesssim(1+B(\tau,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac12\right)p-\frac{s-1}{2}}\|u\|^p_{X(t)}.$$

Then we have

$$\||u(\tau,x)|^p\|_{\dot H^{s-1}(\mathbb{R}^n)}\lesssim(1+B(\tau,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac12\right)p-\frac{s-1}{2}}\|u\|^p_{X(t)}. \tag{1.96}$$

We remark that $n\ge4$ implies $s>3$ which leads to

$$-\frac{n}{2}\Big(\frac{1}{m}-\frac12\Big)p-\frac{s-1}{2}>-\frac{n}{2m}p+\frac{n}{2m}>-\frac{n}{2m}p+\frac{n}{4}. \tag{1.97}$$

Now we estimate $\||D|^{s-1}u_t^{nl}(t,\cdot)\|_{L^2(\mathbb{R}^n)}$. We have

$$\big\||D|^{s-1}u_t^{nl}(t,\cdot)\big\|_{L^2(\mathbb{R}^n)}\lesssim\int_0^t\big\||D|^{s-1}\partial_tE_1(t,\tau,x)*_{(x)}|u(\tau,x)|^p\big\|_{L^2(\mathbb{R}^n)}d\tau.$$

As before we divide the integral into two parts. For the fist part $\tau\in[0,\frac{t}{2}]$ after using (1.27) we get

$$\begin{aligned}&\int_0^{\frac t2}b(t)^{-1}b(\tau)^{-1}(1+B(t,\tau))^{-\frac{n}{2}\left(\frac1m-\frac12\right)-1-\frac{s-1}{2}}\||u(\tau,x)|^p\|_{L^m(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)\cap\dot H^{s-1}(\mathbb{R}^n)}d\tau\\ &\quad\lesssim\|u\|^p_{X(t)}\int_0^{\frac t2}b(t)^{-1}b(\tau)^{-1}(1+B(t,\tau))^{-\frac{n}{2}\left(\frac1m-\frac12\right)-1-\frac{s-1}{2}}(1+B(\tau,0))^{-\frac{n}{2}\left(\frac1m-\frac12\right)p-\frac{s-1}{2}}d\tau\\ &\quad\lesssim\|u\|^p_{X(t)}b(t)^{-1}(1+B(t,0))^{-\frac{n}{2}\left(\frac1m-\frac12\right)-1-\frac{s-1}{2}}\int_0^{\frac t2}b(\tau)^{-1}(1+B(\tau,0))^{-\frac{n}{2}\left(\frac1m-\frac12\right)p-\frac{s-1}{2}}d\tau\\ &\quad\lesssim\|u\|^p_{X(t)}b(t)^{-1}(1+B(t,0))^{-\frac{n}{2}\left(\frac1m-\frac12\right)-1-\frac{s-1}{2}}\int_0^{\frac t2}b(\tau)^{-1}(1+B(\tau,0))^{-\frac{n}{2}\left(\frac1m-\frac12\right)p-1}d\tau\\ &\quad\lesssim\|u\|^p_{X(t)}b(t)^{-1}(1+B(t,0))^{-\frac{n}{2}\left(\frac1m-\frac12\right)-1-\frac{s-1}{2}},\end{aligned}$$

where we use $s>3$ and $p>\frac{2m}{n}\big(\frac{2}{2-m}\big)+1$.

For $\tau \in [\frac{t}{2}, t]$, by using the estimate (1.27) with $m = 2$, we derive the estimate

$$\begin{aligned}
&\int_{\frac{t}{2}}^{t} b(t)^{-1} b(\tau)^{-1}(1+B(t,\tau))^{-1-\frac{s-1}{2}} \||u(\tau,x)|^p\|_{L^2(\mathbb{R}^n)\cap \dot{H}^{s-1}(\mathbb{R}^n)} d\tau \\
&\quad \lesssim \|u\|_{X(t)}^p \int_{\frac{t}{2}}^{t} b(t)^{-1} b(\tau)^{-1}(1+B(t,\tau))^{-1-\frac{s-1}{2}}(1+B(\tau,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)p-\frac{s-1}{2}} d\tau \\
&\quad \lesssim \|u\|_{X(t)}^p b(t)^{-1}(1+B(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)p-\frac{s-1}{2}} \\
&\quad \lesssim \|u\|_{X(t)}^p b(t)^{-1}(1+B(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-1-\frac{s-1}{2}}
\end{aligned}$$

for $p > \frac{2m}{n}\left(\frac{2}{2-m}\right)+1$ which is supposed in (1.90). Consequently, we get

$$\||D|^{s-1} u_t^{nl}(t,\cdot)\|_{L^2(\mathbb{R}^n)} \lesssim \|u\|_{X(t)}^p b(t)^{-1}(1+B(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-1-\frac{s-1}{2}}. \tag{1.98}$$

To complete the proof we mention that in the same way one can prove the estimates

$$\begin{aligned}
\|u^{nl}(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \|u\|_{X(t)}^p (1+B(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)}, && (1.99)\\
\|u_t^{nl}(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \|u\|_{X(t)}^p b(t)^{-1}(1+B(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-1}, && (1.100)\\
\||D|^s u^{nl}(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \|u\|_{X(t)}^p (1+B(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s}{2}}. && (1.101)
\end{aligned}$$

From (1.93) and (1.98) to (1.101) we obtain the first inequality (1.91).
Now we prove (1.92). From the definition of the solution space $X(t)$ and its norm we can use the inequalities (1.71) and (1.72) from the proof of Theorem 1.10. The modification to this proof is the estimate of

$$\big\||u(\tau,x)|^p - |v(\tau,x)|^p\big\|_{\dot{H}^{s-1}(\mathbb{R}^n)}.$$

Due to the fractional Leibniz rule in form of Proposition A.2 we have the estimate

$$\begin{aligned}
&\big\||u(\tau,x)|^p - |v(\tau,x)|^p\big\|_{\dot{H}^{s-1}(\mathbb{R}^n)} = \big\||D|^{s-1}(|u(\tau,x)|^p - |v(\tau,x)|^p)\big\|_{L^2(\mathbb{R}^n)} \\
&\quad \lesssim \int_0^1 \big\||D|^{s-1}\left\{(u-v)(u-r(u-v))|u-r(u-v)|^{p-2}\right\}\big\|_{L^2(\mathbb{R}^n)} dr \\
&\quad \lesssim \int_0^1 \big\||D|^{s-1}(u-v)\big\|_{L^{q_1}(\mathbb{R}^n)} \big\|(u-r(u-v))|u-r(u-v)|^{p-2}\big\|_{L^{q_2}(\mathbb{R}^n)} dr \\
&\qquad + \int_0^1 \big\|u-v\big\|_{L^{q_3}(\mathbb{R}^n)} \big\||D|^{s-1}\left[(u-r(u-v))|u-r(u-v)|^{p-2}\right]\big\|_{L^{q_4}(\mathbb{R}^n)} dr,
\end{aligned}$$

where

$$\frac{1}{q_1}+\frac{1}{q_2}=\frac{1}{q_3}+\frac{1}{q_4}=\frac{1}{2}.$$

Using the fractional Gagliardo-Nirenberg inequality gives

$$\begin{aligned}
&\big\||D|^{s-1}(u-v)\big\|_{L^{q_1}(\mathbb{R}^n)} \\
&\quad \lesssim \big\|(u-v)\big\|_{L^2(\mathbb{R}^n)}^{1-\theta_1} \big\||D|^s(u-v)\big\|_{L^2(\mathbb{R}^n)}^{\theta_1} \\
&\quad \lesssim (1+B(\tau,0))^{(1-\theta_1)\left(-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)\right)+\theta_1\left(-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s}{2}\right)} \big\|u-v\big\|_{X(t)} \\
&\quad \lesssim (1+B(\tau,0))^{-\frac{n}{2m}-\frac{s-1}{2}+\frac{n}{2q_1}} \big\|u-v\big\|_{X(t)},
\end{aligned}$$

where
$$\theta_1=\frac{n}{s}\Big(\frac{1}{2}-\frac{1}{q_1}+\frac{s-1}{n}\Big)\in\Big[\frac{s-1}{s},1\Big]$$
with a parameter q_1 satisfying the condition $\frac{n-2}{2n}\leq\frac{1}{q_1}\leq\frac{1}{2}$. Then,
$$\big\||D|^{s-1}(u-v)\big\|_{L^{q_1}(\mathbb{R}^n)}\lesssim(1+B(\tau,0))^{-\frac{n}{2m}-\frac{s-1}{2}+\frac{n}{2q_1}}\big\|u-v\big\|_{X(t)}. \tag{1.102}$$
We use the classical Gagliardo-Nirenberg inequality for estimating the second term. Hence, it follows
$$\begin{aligned}
&\big\|(u-r(u-v))|u-r(u-v)|^{p-2}\big\|_{L^{q_2}(\mathbb{R}^n)}\\
&\quad\lesssim\big\|u-r(u-v)\big\|^{p-1}_{L^{q_2(p-1)}(\mathbb{R}^n)}\\
&\quad\lesssim\big\|u-r(u-v)\big\|^{(1-\theta_2)(p-1)}_{L^2(\mathbb{R}^n)}\big\||D|^s\left(u-r(u-v)\right)\big\|^{\theta_2(p-1)}_{L^2(\mathbb{R}^n)}\\
&\quad\lesssim(1+B(\tau,0))^{(1-\theta_2)(p-1)\left(-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)\right)+\theta_2(p-1)\left(-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s}{2}\right)}\big\|u-r(u-v)\big\|^{p-1}_{X(t)}\\
&\quad\lesssim(1+B(\tau,0))^{-\frac{n}{2m}(p-1)+\frac{n}{2q_2}}\big\|u-r(u-v)\big\|^{p-1}_{X(t)},
\end{aligned}$$
where $\theta_2=\frac{n}{s}\big(\frac{1}{2}-\frac{1}{q_2(p-1)}\big)\in[0,1]$ with a parameter q_2 satisfying the condition $\frac{1}{q_2}\leq\frac{p-1}{2}$. Then,
$$\big\|u-r(u-v)\big\|^{p-1}_{L^{q_2}(\mathbb{R}^n)}\lesssim(1+B(\tau,0))^{-\frac{n}{2m}(p-1)+\frac{n}{2q_2}}\big\|u-r(u-v)\big\|^{p-1}_{X(t)}. \tag{1.103}$$
If we choose $\frac{1}{q_1}=\frac{n-2}{2n}$, then we get $\frac{1}{q_2}=\frac{1}{n}$. So, we can verify the condition on θ_2 after taking account of (1.90). Finally, from (1.102) and (1.103) we get
$$\begin{aligned}
&\big\||D|^{s-1}(u-v)\big\|_{L^{q_1}(\mathbb{R}^n)}\big\|(u-r(u-v))|u-r(u-v)|^{p-2}\big\|_{L^{q_2}(\mathbb{R}^n)}\\
&\qquad\lesssim(1+B(\tau,0))^{-\frac{n}{2m}p+\frac{n}{4}-\frac{s-1}{2}}\big\|u-v\big\|_{X(t)}\big\|u-r(u-v)\big\|^{p-1}_{X(t)}.
\end{aligned}\tag{1.104}$$
Now we estimate the second integral. After using the classical Gagliardo-Nirenberg inequality we estimate the first term as follows:
$$\begin{aligned}
\big\|u-v\big\|_{L^{q_3}(\mathbb{R}^n)}&\lesssim\big\|u-v\big\|^{1-\theta_3}_{L^2(\mathbb{R}^n)}\big\||D|^s(u-v)\big\|^{\theta_3}_{L^2(\mathbb{R}^n)}\\
&\lesssim(1+B(\tau,0))^{(1-\theta_3)\left(-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)\right)+\theta_3\left(-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s}{2}\right)}\big\|u-v\big\|_{X(t)}\\
&\lesssim(1+B(\tau,0))^{-\frac{n}{2m}+\frac{n}{2q_3}}\big\|u-v\big\|_{X(t)},
\end{aligned}$$
where $\theta_3=\frac{n}{s}\big(\frac{1}{2}-\frac{1}{q_3}\big)\in[0,1]$ for $\frac{1}{q_3}\leq\frac{1}{2}$. So,
$$\big\|u-v\big\|_{L^{q_3}(\mathbb{R}^n)}\lesssim(1+B(\tau,0))^{-\frac{n}{2m}+\frac{n}{2q_3}}\big\|u-v\big\|_{X(t)}. \tag{1.105}$$
To estimate the second term, we follow the same approach as we applied to estimate $\big\||u(t,\cdot)|^p\big\|_{\dot{H}^{s-1}}$. This means, we apply again Corollary A.3. In this way we obtain
$$\begin{aligned}
&\big\||D|^{s-1}\left[(u-r(u-v))|u-r(u-v)|^{p-2}\right]\big\|_{L^{q_4}(\mathbb{R}^n)}\\
&\qquad=\big\|(u-r(u-v))|u-r(u-v)|^{p-2}\big\|_{\dot{H}^{s-1}_{q_4}(\mathbb{R}^n)}\\
&\qquad\lesssim\big\|u-r(u-v)\big\|_{\dot{H}^{s-1}_{q_4}(\mathbb{R}^n)}\big\|u-r(u-v)\big\|^{p-2}_{L^\infty(\mathbb{R}^n)}.
\end{aligned}$$

Using the fractional Gagliardo-Nirenberg inequality leads to

$$\big\|u-r(u-v)\big\|_{\dot H^{s-1}_{q_4}(\mathbb{R}^n)} \lesssim (1+B(\tau,0))^{-\frac{n}{2m}+\frac{n}{2q_4}-\frac{s-1}{2}}\big\|u-r(u-v)\big\|_{X(t)},$$

where $\theta_4=\frac{n}{s}\big(\frac{1}{2}-\frac{1}{q_4}\big)+\frac{s-1}{s}\in[\frac{s-1}{s},1]$. Furthermore, we have

$$\big\|u-r(u-v)\big\|^{p-2}_{L^2(\mathbb{R}^n)} \lesssim (1+B(\tau,0))^{-\frac{n}{2}\big(\frac{1}{m}-\frac{1}{2}\big)(p-2)}\big\|u-r(u-v)\big\|^{p-2}_{X(t)},$$

$$\big\|u-r(u-v)\big\|^{p-2}_{\dot H^{s-1}(\mathbb{R}^n)} \lesssim (1+B(\tau,0))^{\big(-\frac{n}{2}\big(\frac{1}{m}-\frac{1}{2}\big)-\frac{s-1}{2}\big)(p-2)}\big\|u-r(u-v)\big\|^{p-2}_{X(t)}.$$

Finally, we get

$$\begin{aligned}&\||D|^{s-1}\left[(u-r(u-v))|u-r(u-v)|^{p-2}\right]\|_{L^{q_4}(\mathbb{R}^n)}\\ &\quad\lesssim (1+B(\tau,0))^{-\frac{n}{2m}(p-1)+\frac{n}{2q_4}-\frac{s-1}{2}+\frac{n}{4}(p-2)}\big\|u-r(u-v)\big\|^{p-1}_{X(t)}.\end{aligned} \tag{1.106}$$

Summarizing gives the desired estimate

$$\begin{aligned}&\big\|u-v\big\|_{L^{q_3}(\mathbb{R}^n)}\big\||D|^{s-1}\left[(u-r(u-v))|u-r(u-v)|^{p-2}\right]\big\|_{L^{q_4}(\mathbb{R}^n)}\\ &\quad\lesssim (1+B(\tau,0))^{-\frac{n}{2m}p+\frac{n}{4}(p-1)-\frac{s-1}{2}}\big\|u-v\big\|_{X(t)}\big\|u-r(u-v)\big\|^{p-1}_{X(t)}.\end{aligned} \tag{1.107}$$

All together implies for $0\le\tau\le t$ the estimate

$$\begin{aligned}&\big\||u(\tau,x)|^p-|v(\tau,x)|^p\big\|_{\dot H^{s-1}(\mathbb{R}^n)}\\ &\quad\lesssim (1+B(\tau,0))^{-\frac{n}{2m}p+\frac{n}{4}(p-1)-\frac{s-1}{2}}\big\|u-v\big\|_{X(t)}\big(\|u\|^{p-1}_{X(t)}+\|v\|^{p-1}_{X(t)}\big).\end{aligned} \tag{1.108}$$

To get (1.92) we follow the same steps of the proof of (1.91) using (1.108) and taking into consideration

$$-\frac{n}{2m}p+\frac{n}{4}(p-1)-\frac{s-1}{2}\le-\frac{n}{2}\Big(\frac{1}{m}-\frac{1}{2}\Big)p-\frac{s-1}{2}.$$

The proof is completed. □

Example 1.6. *In the second statement of Example 1.3, where we have chosen $m=\frac{7}{5}$ and $n=6$, we have $\max\{s;\frac{2m}{n}(\frac{2}{2-m})+1\}=s$. This implies in our case, where the data are supposed to possess a large regularity, an unbounded admissible range for p with respect to the regularity, that is $p>s$.*

Corollary 1.1. *Let us assume $n\ge4$. The data are supposed to satisfy the following condition:*

$$(u_0,u_1)\in\mathcal{A}_{1,s}=\big(H^s(\mathbb{R}^n)\cap L^1(\mathbb{R}^n)\big)\times\big(H^{s-1}(\mathbb{R}^n)\cap L^1(\mathbb{R}^n)\big),\ \ s>\frac{n}{2}+1.$$

Finally, let $p>s$. Then, there exists a small constant ϵ_0 such that if

$$\|(u_0,u_1)\|_{\mathcal{A}_{1,s}}\le\epsilon_0,$$

then there exists a uniquely determined globally (in time) energy solution to (1.89) which belongs to the function space

$$\mathcal{C}\big([0,\infty),H^s(\mathbb{R}^n)\big)\cap\mathcal{C}^1\big([0,\infty),H^{s-1}(\mathbb{R}^n)\big).$$

The solution satisfies the estimates:

$$\begin{aligned}
\|u(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \big(1+B(t,0)\big)^{-\frac{n}{4}}\|(u_0,u_1)\|_{\mathcal{A}_{1,s}},\\
\|u_t(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim b(t)^{-1}\big(1+B(t,0)\big)^{-\frac{n}{4}-1}\|(u_0,u_1)\|_{\mathcal{A}_{1,s}},\\
\||D|^s u(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \big(1+B(t,0)\big)^{-\frac{n}{4}-\frac{s}{2}}\|(u_0,u_1)\|_{\mathcal{A}_{1,s}},\\
\||D|^{s-1} u_t(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim b(t)^{-1}\big(1+B(t,0)\big)^{-\frac{n}{4}-\frac{s-1}{2}-1}\|(u_0,u_1)\|_{\mathcal{A}_{1,s}}.
\end{aligned}$$

We observe from the proof of Theorem 1.12 that the condition $p > \frac{2m}{n}\left(\frac{2}{2-m}\right)+1$ appears by estimating the norm $\||D|^{s-1}u_t^{nl}(t,\cdot)\|_{L^2(\mathbb{R}^n)}$, in particular, the integral for $\tau\in\left[\frac{t}{2},t\right]$. As we did in the case where the data are supposed to belong to Sobolev spaces with suitable regularity we may apply Lemma 1.2 to get the following result.

Theorem 1.13. *Let us assume $n\geq 3$. The data (u_0,u_1) are supposed to belong to*

$$\mathcal{A}_{m,s,\varepsilon} = \big(H^s(\mathbb{R}^n)\cap L^m(\mathbb{R}^n)\big)\times\big(H^{s-1+\varepsilon}(\mathbb{R}^n)\cap L^m(\mathbb{R}^n)\big),\;\; s>\frac{n}{2}+1.$$

Finally, the exponent p satisfies the condition

$$p > \max\Big\{s;\frac{2m}{n}\Big(\frac{2}{2-m}\Big)+1\Big\}. \tag{1.109}$$

Then, there exists a small positive constant ϵ_0 such that if

$$\|(u_0,u_1)\|_{\mathcal{A}_{m,s,\varepsilon}}\leq\epsilon_0,$$

then there exists a uniquely determined globally (in time) energy solution to (1.89) in

$$\mathcal{C}\big([0,\infty),H^s(\mathbb{R}^n)\big)\cap\mathcal{C}^1\big([0,\infty),H^{s-1+\varepsilon}(\mathbb{R}^n)\big).$$

The solution satisfies the following estimates:

$$\begin{aligned}
\|u(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \big(1+B(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)}\|(u_0,u_1)\|_{\mathcal{A}_{m,s,\varepsilon}},\\
\|u_t(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim b(t)^{-1}\big(1+B(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-1}\|(u_0,u_1)\|_{\mathcal{A}_{m,s,\varepsilon}},\\
\||D|^s u(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \big(1+B(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s}{2}}\|(u_0,u_1)\|_{\mathcal{A}_{m,s,\varepsilon}},\\
\||D|^{s-1} u_t(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim b(t)^{-1}\big(1+B(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s-1}{2}-1}\|(u_0,u_1)\|_{\mathcal{A}_{m,s,\varepsilon}}.
\end{aligned}$$

Proof. To prove this theorem we follow the same steps of the proof of Theorem 1.12. The only modification is the use of Lemma 1.2 to estimate the term $\||D|^{s-1}u_t^{nl}(t,\cdot)\|_{L^2(\mathbb{R}^n)}$, in particular, the integral for $\tau\in\left[\frac{t}{2},t\right]$. Indeed, we have

$$\begin{aligned}
&\int_{\frac{t}{2}}^t \Big\||D|^{s-1}\partial_t E_1(t,\tau,x)\ast_{(x)}|u(\tau,x)|^p\Big\|_{L^2(\mathbb{R}^n)}d\tau\\
&\quad\lesssim \int_{\frac{t}{2}}^t b(t)^{-1}b(\tau)^{-1}B(t,\tau)^{-1+\frac{\varepsilon}{2}}\||u(\tau,x)|^p\|_{\dot{H}^{s-1}(\mathbb{R}^n)\cap\dot{H}^{s-1+\varepsilon}(\mathbb{R}^n)}d\tau.
\end{aligned} \tag{1.110}$$

Similarly to (1.96), after using fractional powers from Corollary A.3 for $0\leq\tau\leq t$, gives

$$\begin{aligned}
\||u(\tau,x)|^p\|_{\dot{H}^{s-1}(\mathbb{R}^n)} &\lesssim (1+B(\tau,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)p-\frac{s-1}{2}}\|u\|_{X(t)}^p,\\
\||u(\tau,x)|^p\|_{\dot{H}^{s-1+\varepsilon}(\mathbb{R}^n)} &\lesssim (1+B(\tau,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)p-\frac{s-1}{2}-\frac{\varepsilon}{2}}\|u\|_{X(t)}^p.
\end{aligned}$$

Consequently, we have

$$\begin{aligned}
&\int_{\frac{t}{2}}^{t} \big\| |D|^{s-1}\partial_t E_1(t,\tau,x) \ast_{(x)} |u(\tau,x)|^p \big\|_{L^2(\mathbb{R}^n)} d\tau \\
&\quad \lesssim \|u\|_{X(t)}^p \int_{\frac{t}{2}}^{t} b(t)^{-1} b(\tau)^{-1} B(t,\tau)^{-1+\frac{\varepsilon}{2}} (1+B(\tau,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)p-\frac{s-1}{2}} d\tau \\
&\quad \lesssim \|u\|_{X(t)}^p b(t)^{-1} (1+B(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-1-\frac{s-1}{2}}
\end{aligned}$$

for $p > \frac{2m}{n}\big(\frac{2}{2-m}\big)+1$ which is supposed in (1.109). In this way we complete the proof. $\square$

We remark that the use of Lemma 1.2 does not bring any benefit to the admissible range of p and, furthermore, we require more regularity for the data. So, this seems to be not the best way to estimate $\||D|^{s-1}u_t^{nl}(t,\cdot)\|_{L^2(\mathbb{R}^n)}$, in particular, the integral for $\tau \in \big[\frac{t}{2}, t\big]$. In addition to Lemma 1.2 we try to modify the estimate of $\|u(\tau,\cdot)\|_{L^\infty(\mathbb{R}^n)}$ by using Lemma A.6 which is introduced by D'Abbicco, Ebert and Lucente in [7]. Then we get for $s^* < \frac{n}{2}$ the following estimate:

$$\begin{aligned}
\||u(\tau,x)|^p\|_{\dot{H}^{s-1}(\mathbb{R}^n)} &\lesssim \|u(\tau,\cdot)\|_{\dot{H}^{s-1}(\mathbb{R}^n)} \|u(\tau,\cdot)\|_{L^\infty(\mathbb{R}^n)}^{p-1} \\
&\lesssim \|u(\tau,\cdot)\|_{\dot{H}^{s-1}(\mathbb{R}^n)} \big(\|u(\tau,\cdot)\|_{\dot{H}^{s^*}(\mathbb{R}^n)} + \|u(\tau,\cdot)\|_{\dot{H}^{s-1}(\mathbb{R}^n)}\big)^{p-1} \\
&\lesssim \|u(\tau,\cdot)\|_{\dot{H}^{s-1}(\mathbb{R}^n)}^p + \|u(\tau,\cdot)\|_{\dot{H}^{s-1}(\mathbb{R}^n)} \|u(\tau,\cdot)\|_{\dot{H}^{s^*}(\mathbb{R}^n)}^{p-1}.
\end{aligned}$$

Using the fractional Gagliardo-Nirenberg inequality and the definition of the solution space $X(t)$ we obtain for $0 \le \tau \le t$

$$\||u(\tau,x)|^p\|_{\dot{H}^{s-1}(\mathbb{R}^n)} \lesssim (1+B(\tau,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)p-\frac{s-1}{2}-\frac{s^*}{2}(p-1)} \|u\|_{X(t)}^p. \tag{1.111}$$

Similarly, we can get

$$\||u(\tau,x)|^p\|_{\dot{H}^{s-1+\varepsilon}(\mathbb{R}^n)} \lesssim (1+B(\tau,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)p-\frac{s-1}{2}-\frac{\varepsilon}{2}-\frac{s^*}{2}(p-1)} \|u\|_{X(t)}^p. \tag{1.112}$$

Plugging the last estimates into the above estimate and combining it with the steps of the proof of Theorem 1.12 we obtain the following result.

Theorem 1.14. *Let us assume $n \ge 3$. The data satisfy the condition*

$$(u_0,u_1) \in \mathcal{A}_{m,s,\varepsilon} = \big(H^s(\mathbb{R}^n) \cap L^m(\mathbb{R}^n)\big) \times \big(H^{s-1+\varepsilon}(\mathbb{R}^n) \cap L^m(\mathbb{R}^n)\big), \quad s > \frac{n}{2}+1.$$

Finally, the exponent p satisfies the condition

$$p > \max\Big\{s; \frac{2}{n\left(\frac{2-m}{2m}\right)-s^*}+1\Big\}, \tag{1.113}$$

where $s^* < \frac{n}{2}\left(\frac{2-m}{m}\right)$. Then, there exists a uniquely determined globally (in time) energy solution to (1.89) in

$$\mathcal{C}\big([0,\infty), H^s(\mathbb{R}^n)\big) \cap \mathcal{C}^1\big([0,\infty), H^{s-1+\varepsilon}(\mathbb{R}^n)\big).$$

The solution satisfies the following estimates:

$$\begin{aligned}
\|u(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \big(1+B(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)}\|(u_0,u_1)\|_{\mathcal{A}_{m,s,\varepsilon}},\\
\|u_t(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim b(t)^{-1}\big(1+B(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-1}\|(u_0,u_1)\|_{\mathcal{A}_{m,s,\varepsilon}},\\
\||D|^s u(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \big(1+B(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s}{2}}\|(u_0,u_1)\|_{\mathcal{A}_{m,s,\varepsilon}},\\
\||D|^{s-1} u_t(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim b(t)^{-1}\big(1+B(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s-1}{2}-1}\|(u_0,u_1)\|_{\mathcal{A}_{m,s,\varepsilon}}.
\end{aligned}$$

Remark 1.5. *We can classify the tools which are used to estimate the norm $\||D|^{s-1}u_t^{nl}(t,\cdot)\|_{L^2(\mathbb{R}^n)}$, in particular, the integral for $\tau \in \left[\frac{t}{2},t\right]$ in the proofs of the previous theorems as follows:*

- *In the proof of Theorem 1.12 we used a $H^{s-1}(\mathbb{R}^n)-\dot{H}^{s-1}(\mathbb{R}^n)$ estimate for the solutions to a family of parameter-dependent Cauchy problems, more precisely, the estimate (1.27), and the estimate $\|u\|_{L^\infty(\mathbb{R}^n)} \le \|u\|_{L^2(\mathbb{R}^n)} + \|u\|_{\dot{H}^s(\mathbb{R}^n)}$.*
- *In the proof of Theorem 1.13 we used Lemma 1.2 and the estimate $\|u\|_{L^\infty(\mathbb{R}^n)} \le \|u\|_{L^2(\mathbb{R}^n)} + \|u\|_{\dot{H}^{s-1}(\mathbb{R}^n)}$.*
- *In Theorem 1.14 we used Lemma 1.2 and the estimate $\|u\|_{L^\infty(\mathbb{R}^n)} \le \|u\|_{\dot{H}^{s^*}(\mathbb{R}^n)} + \|u\|_{\dot{H}^{s-1}(\mathbb{R}^n)}$ from Proposition A.6.*

In the following theorem we will use instead a $H^{s-1}(\mathbb{R}^n)-\dot{H}^{s-1}(\mathbb{R}^n)$ estimate for solutions to a family of parameter-dependent Cauchy problems, more precisely, the estimate (1.27) and the estimate from Proposition A.6.

Theorem 1.15. *Let us assume $n \ge 4$. The data satisfy the condition*

$$(u_0,u_1) \in \mathcal{A}_{m,s} = \big(H^s(\mathbb{R}^n)\cap L^m(\mathbb{R}^n)\big)\times\big(H^{s-1}(\mathbb{R}^n)\cap L^m(\mathbb{R}^n)\big),\ \ s > \frac{n}{2}+1.$$

Finally, the exponent p satisfies the condition

$$p > s. \tag{1.114}$$

Then, there exists a small positive constant ϵ_0 such that if

$$\|(u_0,u_1)\|_{\mathcal{A}_{m,s}} \le \epsilon_0,$$

then there exists a uniquely determined globally (in time) energy solution to (1.89) in

$$\mathcal{C}\big([0,\infty),H^s(\mathbb{R}^n)\big)\cap\mathcal{C}^1\big([0,\infty),H^{s-1}(\mathbb{R}^n)\big).$$

The solution satisfies the following estimates:

$$\begin{aligned}
\|u(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \big(1+B(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)}\|(u_0,u_1)\|_{\mathcal{A}_{m,s}},\\
\|u_t(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim b(t)^{-1}\big(1+B(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-1}\|(u_0,u_1)\|_{\mathcal{A}_{m,s}},\\
\||D|^s u(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \big(1+B(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s}{2}}\|(u_0,u_1)\|_{\mathcal{A}_{m,s}},\\
\||D|^{s-1} u_t(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim b(t)^{-1}\big(1+B(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s-1}{2}-1}\|(u_0,u_1)\|_{\mathcal{A}_{m,s}}.
\end{aligned}$$

Proof. As we did in the proofs to the Theorems 1.13 and 1.14, to complete the proof we need just to modify the estimate of the "most complicate" norm in the norm of the solution space $X(t)$ which is the norm $\||D|^{s-1}u_t^{nl}(t,\cdot)\|_{L^2(\mathbb{R}^n)}$, in particular, the integral for $\tau \in \left[\frac{t}{2}, t\right]$. Using the estimate (1.27) with $m=2$ and the estimates (1.95), (1.111) we obtain

$$\begin{aligned}
&\int_{\frac{t}{2}}^{t} \big\||D|^{s-1}\partial_t E_1(t,\tau,x) *_{(x)} |u(\tau,x)|^p\big\|_{L^2(\mathbb{R}^n)} d\tau \\
&\quad \lesssim \int_{\frac{t}{2}}^{t} b(t)^{-1}b(\tau)^{-1}(1+B(t,\tau))^{-1-\frac{s-1}{2}} \||u(\tau,x)|^p\|_{L^2(\mathbb{R}^n)\cap \dot{H}^{s-1}(\mathbb{R}^n)} d\tau \\
&\quad \lesssim \|u\|^p_{X(t)} \int_{\frac{t}{2}}^{t} b(t)^{-1}b(\tau)^{-1}(1+B(t,\tau))^{-1-\frac{s-1}{2}} \\
&\qquad\quad \times \left[(1+B(\tau,0))^{-\frac{n}{2m}p+\frac{n}{4}} + (1+B(\tau,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)p-\frac{s-1}{2}-\frac{s^*}{2}(p-1)}\right] d\tau,
\end{aligned}$$

where $s^* < \frac{n}{2}$. If we choose $s^* = \frac{n}{2} - \varepsilon$, then we get

$$-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)p - \frac{s-1}{2} - \frac{s^*}{2}(p-1) \le -\frac{n}{2m}p + \frac{n}{4}. \tag{1.115}$$

Consequently, we have

$$\begin{aligned}
&\int_{\frac{t}{2}}^{t} \big\||D|^{s-1}\partial_t E_1(t,\tau,x) *_{(x)} |u(\tau,x)|^p\big\|_{L^2(\mathbb{R}^n)} d\tau \\
&\quad \lesssim \|u\|^p_{X(t)} \int_{\frac{t}{2}}^{t} b(t)^{-1}b(\tau)^{-1}(1+B(t,\tau))^{-1-\frac{s-1}{2}}(1+B(\tau,0))^{-\frac{n}{2m}p+\frac{n}{4}} d\tau \\
&\quad \lesssim \|u\|^p_{X(t)} b(t)^{-1}(1+B(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-1-\frac{s-1}{2}},
\end{aligned}$$

where $p > \frac{2m}{n}\left(\frac{s+1}{2}\right) + 1$ which is included in (1.114) by assuming $s > 3$ and $n \ge 4$ similarly to (1.64). In this way we can complete the proof as we did in the proof of Theorem 1.12. □

If we use in the proof of Theorem 1.12 instead of fractional powers the fractional chain rule to estimate $\big\||u(\tau,x)|^p\big\|_{\dot{H}^{s-1}(\mathbb{R}^n)}$ and $\big\||u(\tau,x)|^p - v(\tau,x)|^p\big\|_{\dot{H}^{s-1}(\mathbb{R}^n)}$ which generates a condition $p > \lceil s \rceil$ stronger than (1.114), then we obtain the following result.

Theorem 1.16. *Let us assume $n \ge 4$. The data satisfy the condition*

$$(u_0,u_1) \in \mathcal{A}_{m,s} = \big(H^s(\mathbb{R}^n) \cap L^m(\mathbb{R}^n)\big) \times \big(H^{s-1}(\mathbb{R}^n) \cap L^m(\mathbb{R}^n)\big), \;\; s > \frac{n}{2} + 1.$$

Finally, the exponent p satisfies the condition

$$p > \lceil s \rceil. \tag{1.116}$$

Then, there exists a small positive constant ϵ_0 such that if

$$\|(u_0,u_1)\|_{\mathcal{A}_{m,s}} \le \epsilon_0,$$

then there exists a uniquely determined globally (in time) energy solution to (1.89) in

$$\mathcal{C}\big([0,\infty),H^s(\mathbb{R}^n)\big)\cap\mathcal{C}^1\big([0,\infty),H^{s-1}(\mathbb{R}^n)\big).$$

The solution satisfies the following estimates:

$$\begin{aligned}
\|u(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \big(1+B(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)}\|(u_0,u_1)\|_{\mathcal{A}_{m,s}},\\
\|u_t(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim b(t)^{-1}\big(1+B(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-1}\|(u_0,u_1)\|_{\mathcal{A}_{m,s}},\\
\||D|^s u(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \big(1+B(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s}{2}}\|(u_0,u_1)\|_{\mathcal{A}_{m,s}},\\
\||D|^{s-1}u_t(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim b(t)^{-1}\big(1+B(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s-1}{2}-1}\|(u_0,u_1)\|_{\mathcal{A}_{m,s}}.
\end{aligned}$$

Remark 1.6. *If $s>\frac{n}{2}+1$ is a non-integer positive number, then the statement of Theorem 1.12 gives an admissible range for p which is larger than the one from Theorem 1.16.*

1.6. Final remarks

Let us come back to the Cauchy problem

$$u_{tt}-\Delta u+b(t)u_t=|u|^p,\;\; u(0,x)=u_0(x),\;\; u_t(0,x)=u_1(x), \tag{1.117}$$

where $(t,x)\in[0,\infty)\times\mathbb{R}^n$. In this section we conclude some results which can be obtained from the general results are proved in the previous sections. We distinguish between some global existence results for each dimension by taking the optimal range of admissible p.
From Theorem 1.7 and Theorem 1.9 one can get the following result.

Theorem 1.17. *Let $n\le 2$ and $s\in\left[\frac{n}{2},n\right]\cap[0,1], m\in[1,2)$. Let us assume that the data (u_0,u_1) belong to*

$$\mathcal{A}_{m,s}=\big(H^s(\mathbb{R}^n)\cap L^m(\mathbb{R}^n)\big)\times\big(L^2(\mathbb{R}^n)\cap L^m(\mathbb{R}^n)\big).$$

Finally, let

$$p>p_{Fuj,m}(n).$$

Then, there exists a small constant ϵ_0 such that if

$$\|(u_0,u_1)\|_{\mathcal{A}_{m,s}}\le\epsilon_0,$$

then there exists a uniquely determined globally (in time) Sobolev solution to (1.117) belonging to

$$\mathcal{C}\big([0,\infty),H^s(\mathbb{R}^n)\big)\cap\mathcal{C}^{\lfloor\min\{s;1\}\rfloor}\big([0,\infty),L^2(\mathbb{R}^n)\big).$$

Furthermore, the solution satisfies for $s_0+l=0,s$, where $s_0\in\left[\frac{1}{2},s\right]$, $l=0,1$, the estimates

$$\||D|^{s_0}\partial_t^l u(t,\cdot)\|_{L^2(\mathbb{R}^n)}\lesssim\big(1+B(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-l-\frac{s_0}{2}}(b(t))^{-l}\|(u_0,u_1)\|_{\mathcal{A}_{m,s}}.$$

For large dimension $n \geq 3$ we get from Theorem 1.10, Theorem 1.11 and Theorem 1.15 the following result.

Theorem 1.18. *Let $n \geq 3$ and $(u_0, u_1) \in \mathcal{A}_{m,s,\varepsilon}$, $m \in [1,2)$. The following condition is satisfied for the exponent p:*

$$p > \lceil s \rceil,$$

where

$$\begin{array}{lll} s = \frac{7}{3}, \varepsilon > 0 & \textit{if} & n = 3, \\ s = 3, \varepsilon = 0 & \textit{if} & n \in [4,6], \\ s = \frac{n}{2}, \varepsilon = 0 & \textit{if} & n \geq 7. \end{array}$$

Then, there exists a small constant ϵ_0 such that if

$$\|(u_0, u_1)\|_{\mathcal{A}_{m,s,\varepsilon}} \leq \epsilon_0,$$

then there exists a uniquely determined globally (in time) energy solution to (1.117) in

$$\mathcal{C}\big([0,\infty), H^s(\mathbb{R}^n)\big) \cap \mathcal{C}^1\big([0,\infty), H^{s-1+\varepsilon}(\mathbb{R}^n)\big).$$

Furthermore, the solution satisfies the estimates

$$\begin{aligned} \|u(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \big(1 + B(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)} \|(u_0,u_1)\|_{\mathcal{A}_{m,s,\varepsilon}}, \\ \|u_t(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim b(t)^{-1}\big(1 + B(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-1} \|(u_0,u_1)\|_{\mathcal{A}_{m,s,\varepsilon}}, \\ \||D|^s u(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \big(1 + B(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s}{2}} \|(u_0,u_1)\|_{\mathcal{A}_{m,s,\varepsilon}}, \\ \||D|^{s-1} u_t(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim b(t)^{-1}\big(1 + B(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s-1}{2}-1} \|(u_0,u_1)\|_{\mathcal{A}_{m,s,\varepsilon}}. \end{aligned}$$

Remark 1.7. *For $n \geq 7$ we remark that the admissible range for p is defined in Theorem 1.10 for data belonging to Sobolev spaces with suitable regularity. In other words, $p > \lceil s \rceil = \lceil \frac{n}{2} \rceil$ because $\lceil \frac{n}{2} \rceil \leq \frac{n}{2} + 1$ if we want to apply Theorem 1.15.*

2. Weakly coupled systems of semilinear classical damped waves with different power nonlinearities

In this chapter, we apply the results of the first chapter to study weakly coupled systems of semilinear damped waves with the same time-dependent coefficients in the dissipation terms, that is, we concern with the following Cauchy problem for a weakly coupled system of semilinear damped wave equations

$$\begin{aligned} &u_{tt} - \Delta u + b(t)u_t = |v|^p, \qquad v_{tt} - \Delta v + b(t)v_t = |u|^q, \\ &u(0,x) = u_0(x), u_t(0,x) = u_1(x), v(0,x) = v_0(x), v_t(0,x) = v_1(x), \end{aligned} \tag{2.1}$$

where $t \in [0,\infty), x \in \mathbb{R}^n$ and the damping terms $b(t)u_t$ and $b(t)v_t$ are effective in the sense of [67] and [69] (cf. with Hypothesis 1.1 from Section 1.1).
We study the Cauchy problem (2.1) in several cases with respect to the regularity of data as we did in Chapter 1. Therefore, we recall the following classification of regularity for the data: low regular data, data from energy space, data from Sobolev spaces with suitable regularity and, finally, large regular data.
From Theorems 1.7 and 1.9 we remark that the pivotal condition for the exponent p in the power nonlinearity was defined by the modified Fujita exponent $p_{Fuj,m}(n)$. For this reason we compare in the system (2.1) the exponents p and q with $p_{Fuj,m}(n)$. In the case where only one exponent is above $p_{Fuj,m}(n)$ we shall prove a global (in time) existence result with a loss of decay and the following interaction condition

$$\frac{n}{2} > m\Big(\frac{\max\{p;q\}+1}{pq-1}\Big). \tag{2.2}$$

This condition also appeared in previous results, where the coefficients of dissipation terms were constants and equal to 1 (see the papers [40, 60] and [45]). But, in our case we feel the effect of an additional regularity $L^m(\mathbb{R}^n)$, $m \in [1,2)$. At the end of Sections 2.1 and 2.2 we will show some benefits of taking the data with different additional regularities, namely, we suppose $(u_0,u_1) \in \mathcal{A}_{m_1,s}$ and $(v_0,v_1) \in \mathcal{A}_{m_2,s}$. In the case of high regular data we assume different regularities $(u_0,u_1) \in \mathcal{A}_{m,s_1}$ and $(v_0,v_1) \in \mathcal{A}_{m,s_2}$.

2.1. Low regular data

In this section we are interested in the system (2.1), where the data are taken from the same Sobolev space $H^s(\mathbb{R}^n)$ for $s \in (0,1)$, with the same additional regularity $L^m(\mathbb{R}^n)$. We remark immediately that $p_{Fuj,m}(n) > s$ which means that the regularity has a weak influence on the admissible range for p and q. Therefore, we compare in our statements the exponents p and q of power nonlinearities with the modified Fujita exponent $p_{Fuj,m}(n)$.

2.1.1. Both exponents of power nonlinearities are above the modified Fujita exponent

Now we consider the case that the exponents p and q of the nonlinearities are above the critical exponent which is the modified Fujita exponent. Both equations of the system behave independently like one single equation. For this reason we restrict ourselves to the case $p = q$. Then the case $p \neq q$ can be concluded immediately. So we are interested in the Cauchy problem

$$\begin{aligned} &u_{tt} - \Delta u + b(t)u_t = |v|^p, \qquad v_{tt} - \Delta v + b(t)v_t = |u|^p, \\ &u(0,x) = u_0(x), u_t(0,x) = u_1(x), v(0,x) = v_0(x), v_t(0,x) = v_1(x). \end{aligned} \tag{2.3}$$

Because we do not feel any effect from the coupling, we will present only the result without any proof by using Proposition A.9.

Theorem 2.1. *Let $n \leq \frac{4s}{2-m}$, $n < \frac{2sm}{m-s}$, (u_0,u_1), $(v_0,v_1) \in \mathcal{A}_{m,s} \times \mathcal{A}_{m,s}$, $s \in (0,1)$ and $m \in [1,2)$. Moreover, we suppose*

$$p > p_{Fuj,m}(n) \quad \text{and} \quad \frac{2}{m} \leq p \leq p_{GN,s}(n). \tag{2.4}$$

Then, there exists a small constant ϵ_0 such that if

$$\|(u_0,u_1)\|_{\mathcal{A}_{m,s}} + \|(v_0,v_1)\|_{\mathcal{A}_{m,s}} \leq \epsilon_0,$$

then there exists a uniquely determined globally (in time) Sobolev solution to (2.3) in

$$\left(\mathcal{C}([0,\infty), H^s(\mathbb{R}^n))\right)^2.$$

Furthermore, the solution satisfies the decay estimates

$$\|u(t,\cdot)\|_{L^2(\mathbb{R}^n)} \lesssim \big(1+B(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)} \left(\|(u_0,u_1)\|_{\mathcal{A}_{m,s}} + \|(v_0,v_1)\|_{\mathcal{A}_{m,s}}\right), \tag{2.5}$$

$$\|v(t,\cdot)\|_{L^2(\mathbb{R}^n)} \lesssim \big(1+B(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)} \left(\|(u_0,u_1)\|_{\mathcal{A}_{m,s}} + \|(v_0,v_1)\|_{\mathcal{A}_{m,s}}\right), \tag{2.6}$$

$$\begin{aligned} \big\||D|^s u(t,\cdot)\big\|_{L^2(\mathbb{R}^n)} \lesssim &\big(1+B(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s}{2}} \\ &\times \left(\|(u_0,u_1)\|_{\mathcal{A}_{m,s}} + \|(v_0,v_1)\|_{\mathcal{A}_{m,s}}\right), \end{aligned} \tag{2.7}$$

$$\begin{aligned} \big\||D|^s v(t,\cdot)\big\|_{L^2(\mathbb{R}^n)} \lesssim &\big(1+B(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s}{2}} \\ &\times \left(\|(u_0,u_1)\|_{\mathcal{A}_{m,s}} + \|(v_0,v_1)\|_{\mathcal{A}_{m,s}}\right). \end{aligned} \tag{2.8}$$

Proof. The proof of this theorem is similar to the proof of Theorem 1.7. The only difference appears in the definition of the solution space $X(t)$ and its norm. We introduce

$$X(t) = \big\{(u,v) \in \big(\mathcal{C}([0,t], H^s(\mathbb{R}^n))\big)^2\big\}$$

with the norm

$$\|(u,v)\|_{X(t)} = \sup_{\tau\in[0,t]} \big\{M(\tau,u) + M(\tau,v)\big\},$$

where

$$\begin{aligned}
M(\tau,u) &= \big(1+B(\tau,0)\big)^{\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)}\|u(\tau,\cdot)\|_{L^2(\mathbb{R}^n)} \\
&\quad + \big(1+B(\tau,0)\big)^{\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)+\frac{s}{2}}\||D|^s u(\tau,\cdot)\|_{L^2(\mathbb{R}^n)}, \\
M(\tau,v) &= \big(1+B(\tau,0)\big)^{\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)}\|v(\tau,\cdot)\|_{L^2(\mathbb{R}^n)} \\
&\quad + \big(1+B(\tau,0)\big)^{\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)+\frac{s}{2}}\||D|^s v(\tau,\cdot)\|_{L^2(\mathbb{R}^n)}.
\end{aligned}$$

So, using the conditions on p we can control all components of the norms

$$\|N(u,v)\|_{X(t)} \quad \text{and} \quad \|N(u,v) - N(\tilde{u},\tilde{v})\|_{X(t)}.$$

We apply Proposition A.9 which completes the proof. □

Particular case: If $n = 1$, then the condition (2.4), generated from using of Gagliardo-Nirenberg inequality, will be changed as follows:

$$\begin{aligned}
\tfrac{2}{m} \le p < \infty \quad &\text{if} \quad s \in [\tfrac{1}{2}, 1), \\
\tfrac{2}{m} \le p \le p_{GN,s}(1) \quad &\text{if} \quad s \in (0, \tfrac{1}{2}).
\end{aligned}$$

Remark 2.1. *The table in Example 1.2 and Remark 1.2 remain true for the Cauchy problem (2.3).*

If we have $p \neq q$ and both exponents p and q of the nonlinearities satisfy the condition (2.4), then we obtain a similar result to Theorem 2.1.

Theorem 2.2. *Let $n \le \frac{4s}{2-m}$, $n < \frac{2sm}{m-s}$, (u_0,u_1), $(v_0,v_1) \in \mathcal{A}_{m,s} \times \mathcal{A}_{m,s}$, $s \in (0,1)$ and $m \in [1,2)$. Moreover, we suppose for the exponents p and q the conditions*

$$\min\{p;q\} > p_{Fuj,m}(n) \quad \text{and} \quad \frac{2}{m} \le \min\{p;q\} \le \max\{p;q\} \le p_{GN,s}(n).$$

Then, there exists a small constant ϵ_0 such that if

$$\|(u_0,u_1)\|_{\mathcal{A}_{m,s}} + \|(v_0,v_1)\|_{\mathcal{A}_{m,s}} \le \epsilon_0,$$

then there exists a uniquely determined globally (in time) Sobolev solution to (2.1) in

$$\left(\mathcal{C}([0,\infty), H^s(\mathbb{R}^n))\right)^2.$$

Furthermore, the solution satisfies the same decay estimates (2.5) to (2.8).

2.1.2. Only one exponent is above the modified Fujita exponent

Now we assume that only one power nonlinearity does not satisfy the pivotal requirement of existence in the case of one single equation. This means we assume

$$\min(p,q) \leq p_{Fuj,m}(n) < \max(p,q).$$

At first we consider the case $p<q$.

Remark 2.2. *From the use of the Gagliardo-Nirenberg inequality in the proof of Theorem 1.7 we can not assume that $p,q \notin \big[\frac{2}{m}, \frac{n}{n-2s}\big]$. Then the available way to weaker the condition on p is to suppose $\frac{2}{m} \leq p \leq p_{Fuj,m}(n)$ in order to use $q > p_{Fuj,m}(n)$ to cover this loss.*

Theorem 2.3. *Let $n < \min\{\frac{2m^2}{2-m}; \frac{2sm}{m-s}\}$. The data (u_0,u_1) and (v_0,v_1) are assumed to belong to $\mathcal{A}_{m,s} \times \mathcal{A}_{m,s}$, $s \in (0,1)$ and $m \in [1,2)$. Moreover, let*

$$\frac{2}{m} \leq p < p_{Fuj,m}(n) < q \leq p_{GN,s}(n), \tag{2.9}$$

and

$$\frac{n}{2} > m\Big(\frac{q+1}{pq-1}\Big). \tag{2.10}$$

Then, there exists a small constant ϵ_0 such that if

$$\|(u_0,u_1)\|_{\mathcal{A}_{m,s}} + \|(v_0,v_1)\|_{\mathcal{A}_{m,s}} \leq \epsilon_0,$$

then there exists a uniquely determined globally (in time) Sobolev solution to (2.1) in

$$\big(\mathcal{C}([0,\infty), H^s(\mathbb{R}^n))\big)^2.$$

Furthermore, the solution satisfies the following estimates:

$$\begin{aligned}
\|u(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \big(1+B(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)+\gamma_{n,m}(p)} \left(\|(u_0,u_1)\|_{\mathcal{A}_{m,s}} + \|(v_0,v_1)\|_{\mathcal{A}_{m,s}}\right),\\
\||D|^s u(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \big(1+B(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s}{2}+\gamma_{n,m}(p)} \left(\|(u_0,u_1)\|_{\mathcal{A}_{m,s}} + \|(v_0,v_1)\|_{\mathcal{A}_{m,s}}\right),\\
\|v(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \big(1+B(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)} \left(\|(u_0,u_1)\|_{\mathcal{A}_{m,s}} + \|(v_0,v_1)\|_{\mathcal{A}_{m,s}}\right),\\
\||D|^s v(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \big(1+B(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s}{2}} \left(\|(u_0,u_1)\|_{\mathcal{A}_{m,s}} + \|(v_0,v_1)\|_{\mathcal{A}_{m,s}}\right),
\end{aligned}$$

where $\gamma_{n,m}(p) = -\frac{n}{2m}(p-1)+1$ is the loss of decay in comparison with the corresponding decay estimates for the solution u to the linear Cauchy problem with vanishing right-hand side.

Particular case: If $n=1$, then instead of (2.9) we suppose

$$\begin{aligned}
&\tfrac{2}{m} \leq p < p_{Fuj,m}(1) < q < \infty &&\text{if} \quad s \in \big[\tfrac{1}{2}, 1\big),\\
&\tfrac{2}{m} \leq p < p_{Fuj,m}(1) < q \leq p_{GN,s}(1) &&\text{if} \quad s \in \big(\tfrac{m}{2m+1}, \tfrac{1}{2}\big).
\end{aligned}$$

Remark 2.3. *If we have $p = p_{Fuj,m}(n)$ in condition (2.9), then the restriction of the dimension will change to $n \leq \frac{2m^2}{2-m}$, $n < \frac{2sm}{m-s}$ and an arbitrarily small loss of decay $\gamma_{n,m}(p) = \epsilon$ which is generated by a $\log$ term in t appearing in the control of the nonlinear terms. In the proof we restrict ourselves to the case $p < p_{Fuj,m}(n)$.*

Remark 2.4. *The restriction for the dimension n in the statement of the theorem comes from the admissible range for p and q in condition (2.9).*
We remark that if m is close to 2, then the admissible range for p becomes larger. Conversely, if m is close to 1, then the admissible range for q becomes larger.

Example 2.1. *In the following table we will discuss condition (2.9) which means the admissible range for the exponents p and q of power nonlinearities.*

n	m	Regularity s	Admissible range for p	Admissible range for q
$n=1$	$m \in [1,2)$	$s \in (\frac{m}{2m+1}, \frac{1}{2})$	$\frac{2}{m} < p < 1+2m$	$1+2m < q \leq \frac{1}{1-2s}$
$n=2$	$m \in [1,2)$	$s \in (\frac{m}{m+1}, 1)$	$\frac{2}{m} \leq p < 1+m$	$1+m < q \leq \frac{1}{1-s}$
$n=3$	$m \in (\frac{-3+\sqrt{57}}{4}, 2)$	$s \in (\frac{3m}{2m+3}, 1)$	$\frac{2}{m} \leq p < 1+\frac{2m}{3}$	$1+\frac{2}{3}m < q \leq \frac{3}{3-2s}$
	$m \in [1, \frac{-3+\sqrt{57}}{4}]$	$s \in (0,1)$	empty	
$n=4$	$m \in (\frac{-2+\sqrt{20}}{2}, 2)$	$s \in [\frac{2m}{m+2}, 1)$	$\frac{2}{m} \leq p < 1+\frac{m}{2}$	$1+\frac{m}{2} < q \leq \frac{2}{2-s}$
	$m \in (1, \frac{-2+\sqrt{20}}{2})$	$s \in (0,1)$	empty	
$n=5$	$m \in (\frac{-5+\sqrt{105}}{4}, \frac{5}{3}]$	$s \in (\frac{5m}{2m+5}, 1)$	$\frac{2}{m} < p \leq 1+\frac{2m}{5}$	$1+\frac{2m}{5} < q \leq \frac{5}{5-2s}$
	$m \in [1, \frac{-5+\sqrt{105}}{4})$	$s \in (0,1)$	empty	

The results of the table become useful only if they are combined with condition (2.10). For example, if $n = 1$ and if we choose $s \in [\frac{1}{2}, 1)$, then from the table for $m = \frac{3}{2}$ we get $\frac{4}{3} \leq p \leq 4, 4 < q < \infty$. If we combine this result with condition (2.10), then we obtain a new lower bound $q > 8$ after fixing $p = \frac{7}{2} \in \left[\frac{4}{3}, 4\right]$.

Proof. We define the solution space $X(t)$ by

$$X(t) = \left\{(u,v) \in \left(\mathcal{C}([0,t], H^s(\mathbb{R}^n))\right)^2\right\}$$

with the norm

$$\|(u,v)\|_{X(t)} = \sup_{\tau\in[0,t]} \left\{\left(1+B(\tau,0)\right)^{-\gamma_{n,m}(p)} M(\tau,u) + M(\tau,v)\right\},$$

where

$$\begin{aligned} M(\tau,u) &= \left(1+B(\tau,0)\right)^{\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)} \|u(\tau,\cdot)\|_{L^2(\mathbb{R}^n)} \\ &\quad + \left(1+B(\tau,0)\right)^{\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)+\frac{s}{2}} \||D|^s u(\tau,\cdot)\|_{L^2(\mathbb{R}^n)}, \\ M(\tau,v) &= \left(1+B(\tau,0)\right)^{\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)} \|v(\tau,\cdot)\|_{L^2(\mathbb{R}^n)} \\ &\quad + \left(1+B(\tau,0)\right)^{\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)+\frac{s}{2}} \||D|^s v(\tau,\cdot)\|_{L^2(\mathbb{R}^n)}. \end{aligned}$$

Let N be the mapping on $X(t)$ which is defined by

$$N : (u,v) \in X(t) \to N(u,v) = (u^{ln} + u^{nl}, v^{ln} + v^{nl}), \tag{2.11}$$

where

$$\begin{aligned}
u^{ln}(t,x) &:= E_0(t,0,x) *_{(x)} u_0(x) + E_1(t,0,x) *_{(x)} u_1(x),\\
u^{nl}(t,x) &:= \int_0^t E_1(t,\tau,x) *_{(x)} |v(\tau,x)|^p d\tau,\\
v^{ln}(t,x) &:= E_0(t,0,x) *_{(x)} v_0(x) + E_1(t,0,x) *_{(x)} v_1(x),\\
v^{nl}(t,x) &:= \int_0^t E_1(t,\tau,x) *_{(x)} |u(\tau,x)|^q d\tau.
\end{aligned}$$

The goal is to prove the following estimates:

$$\|N(u,v)\|_{X(t)} \lesssim \|(u_0,u_1)\|_{\mathcal{A}_{m,s}} + \|(v_0,v_1)\|_{\mathcal{A}_{m,s}} + \|(u,v)\|^p_{X(t)} + \|(u,v)\|^q_{X(t)}, \tag{2.12}$$

$$\begin{aligned}
\|N(u,v) - N(\tilde{u},\tilde{v})\|_{X(t)} &\lesssim \|(u,v)-(\tilde{u},\tilde{v})\|_{X(t)}\\
&\times\big(\|(u,v)\|^{p-1}_{X(t)} + \|(\tilde{u},\tilde{v})\|^{p-1}_{X(t)} + \|(u,v)\|^{q-1}_{X(t)} + \|(\tilde{u},\tilde{v})\|^{q-1}_{X(t)}\big).
\end{aligned} \tag{2.13}$$

We begin to prove the first inequality (2.12). We can prove the "linear part" of (2.12) by using the positivity of the loss of decay $\gamma_{n,m}(p)$ and the estimates (1.13), (1.14) from Theorem 1.2. Indeed, we have

$$\begin{aligned}
&\big\|(u^{ln},v^{ln})\big\|_{X(t)}\\
&= \sup_{\tau\in[0,t]}\Big\{\big(1+B(\tau,0)\big)^{\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\gamma_{n,m}(p)}\big\|u^{ln}(\tau,\cdot)\big\|_{L^2(\mathbb{R}^n)}\\
&\quad+\big(1+B(\tau,0)\big)^{\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)+\frac{s}{2}-\gamma_{n,m}(p)}\big\||D|^s u^{ln}(\tau,\cdot)\big\|_{L^2(\mathbb{R}^n)}\\
&\quad+\big(1+B(\tau,0)\big)^{\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)}\big\|v^{ln}(\tau,\cdot)\big\|_{L^2(\mathbb{R}^n)}\\
&\quad+\big(1+B(\tau,0)\big)^{\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)+\frac{s}{2}}\big\||D|^s v^{ln}(\tau,\cdot)\big\|_{L^2(\mathbb{R}^n)}\Big\}\\
&\lesssim \sup_{\tau\in[0,t]}\Big\{\big(1+B(\tau,0)\big)^{\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\gamma_{n,m}(p)}\big(1+B(\tau,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)}\|(u_0,u_1)\|_{\mathcal{A}_{m,s}}\\
&\quad+\big(1+B(\tau,0)\big)^{\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)+\frac{s}{2}}\big(1+B(\tau,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s}{2}-\gamma_{n,m}(p)}\|(u_0,u_1)\|_{\mathcal{A}_{m,s}}\\
&\quad+\big(1+B(\tau,0)\big)^{\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)}\big(1+B(\tau,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)}\|(v_0,v_1)\|_{\mathcal{A}_{m,s}}\\
&\quad+\big(1+B(\tau,0)\big)^{\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)+\frac{s}{2}}\big(1+B(\tau,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s}{2}}\|(v_0,v_1)\|_{\mathcal{A}_{m,s}}\Big\}\\
&\lesssim \|(u_0,u_1)\|_{\mathcal{A}_{m,s}} + \|(v_0,v_1)\|_{\mathcal{A}_{m,s}}.
\end{aligned}$$

Then,

$$\|(u^{ln},v^{ln})\|_{X(t)} \lesssim \|(u_0,u_1)\|_{\mathcal{A}_{m,s}} + \|(v_0,v_1)\|_{\mathcal{A}_{m,s}}. \tag{2.14}$$

For the "nonlinear part" we begin to consider u^{nl}. Using the estimate (1.25) from Theorem 1.5, with $m=2$ for the integral over $[\frac{t}{2},t]$, we get

$$\begin{aligned}
&\big\||D|^s u^{nl}(t,\cdot)\big\|_{L^2(\mathbb{R}^n)}\\
&\lesssim \int_0^{\frac{t}{2}} b(\tau)^{-1}(1+B(t,\tau))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s}{2}}\||v(\tau,x)|^p\|_{L^m(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)}d\tau\\
&\quad+\int_{\frac{t}{2}}^t b(\tau)^{-1}(1+B(t,\tau))^{-\frac{s}{2}}\||v(\tau,x)|^p\|_{L^2(\mathbb{R}^n)}d\tau.
\end{aligned}$$

Similar to (1.34) and (1.35) in the proof of Theorem 1.7 we obtain for $0 \le \tau \le t$ the following estimates:

$$\||v(\tau,x)|^p\|_{L^m(\mathbb{R}^n)} \lesssim (1+B(\tau,0))^{-\frac{n}{2m}p+\frac{n}{2m}}\|(u,v)\|_{X(t)}^p,$$

and

$$\||v(\tau,x)|^p\|_{L^2(\mathbb{R}^n)} \lesssim (1+B(\tau,0))^{-\frac{n}{2m}p+\frac{n}{4}}\|(u,v)\|_{X(t)}^p,$$

provided that $\frac{2}{m} \le p$ and $p \le \frac{n}{n-2s}$ if $n > 2s$ which is included in condition (2.9) due to the use of the Gagliardo-Nirenberg inequality.
For $\tau \in [0, \frac{t}{2}]$ we have

$$\begin{aligned}
&\int_0^{\frac{t}{2}} b(\tau)^{-1}(1+B(t,\tau))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s}{2}}\||v(\tau,x)|^p\|_{L^m(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)}d\tau\\
&\quad\lesssim \|(u,v)\|_{X(t)}^p\int_0^{\frac{t}{2}} b(\tau)^{-1}(1+B(t,\tau))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s}{2}}(1+B(\tau,0))^{-\frac{n}{2m}p+\frac{n}{2m}}d\tau\\
&\quad\lesssim \|(u,v)\|_{X(t)}^p(1+B(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s}{2}}\int_0^{\frac{t}{2}} b(\tau)^{-1}(1+B(\tau,0))^{-\frac{n}{2m}p+\frac{n}{2m}}d\tau\\
&\quad\lesssim (1+B(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s}{2}+\gamma_{n,m}(p)}\|(u,v)\|_{X(t)}^p.
\end{aligned}$$

For $\tau \in [\frac{t}{2}, t]$ we have

$$\begin{aligned}
&\int_{\frac{t}{2}}^{t} b(\tau)^{-1}(1+B(t,\tau))^{-\frac{s}{2}}\||v(\tau,x)|^p\|_{L^2(\mathbb{R}^n)}d\tau\\
&\quad\lesssim \|(u,v)\|_{X(t)}^p\int_{\frac{t}{2}}^{t} b(\tau)^{-1}(1+B(t,\tau))^{-\frac{s}{2}}(1+B(\tau,0))^{-\frac{n}{2m}p+\frac{n}{4}}d\tau\\
&\quad\lesssim \|(u,v)\|_{X(t)}^p(1+B(t,0))^{-\frac{n}{2m}p+\frac{n}{4}}(1+B(t,0))^{1-\frac{s}{2}}\\
&\quad\lesssim (1+B(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s}{2}+\gamma_{n,m}(p)}\|(u,v)\|_{X(t)}^p.
\end{aligned}$$

Consequently, we get

$$\left\||D|^s u^{nl}(t,\cdot)\right\|_{L^2(\mathbb{R}^n)} \lesssim (1+B(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s}{2}+\gamma_{n,m}(p)}\|(u,v)\|_{X(t)}^p. \tag{2.15}$$

Analogously, if $s=0$, then we can prove

$$\left\|u^{nl}(t,\cdot)\right\|_{L^2(\mathbb{R}^n)} \lesssim (1+B(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)+\gamma_{n,m}(p)}\|(u,v)\|_{X(t)}^p. \tag{2.16}$$

For the second component v^{nl} we use the estimate (1.25) from Theorem 1.5, with $m=2$ for the integral over $[\frac{t}{2}, t]$. In this way we obtain

$$\begin{aligned}
&\||D|^s v^{nl}(t,\cdot)\|_{L^2(\mathbb{R}^n)}\\
&\quad\lesssim \int_0^{\frac{t}{2}} b(\tau)^{-1}(1+B(t,\tau))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s}{2}}\||u(\tau,\cdot)|^q\|_{L^m(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)}d\tau\\
&\qquad+\int_{\frac{t}{2}}^{t} b(\tau)^{-1}(1+B(t,\tau))^{-\frac{s}{2}}\||u(\tau,\cdot)|^q\|_{L^2(\mathbb{R}^n)}d\tau.
\end{aligned}$$

In this case, after changing the norm of the solution space with respect to u, the estimates of $\||u(\tau,\cdot)|^q\|_{L^m(\mathbb{R}^n)}$ and $\||u(\tau,\cdot)|^q\|_{L^2(\mathbb{R}^n)}$ will be changed for $0\le\tau\le t$ as follows:

$$\||u(\tau,\cdot)|^q\|_{L^m(\mathbb{R}^n)} \lesssim (1+B(\tau,0))^{-\frac{n}{2m}q+\frac{n}{2m}+\gamma_{n,m}(p)q}\|(u,v)\|^q_{X(t)},$$

and

$$\||u(\tau,\cdot)|^q\|_{L^2(\mathbb{R}^n)} \lesssim (1+B(\tau,0))^{-\frac{n}{2m}q+\frac{n}{4}+\gamma_{n,m}(p)q}\|(u,v)\|^q_{X(t)},$$

where $\frac{2}{m}\le q$ and $q\le\frac{n}{n-2s}$ if $n>2s$ which is included in condition (2.9). This condition is generated from the use of the Gagliardo-Nirenberg inequality.
For $\tau\in[0,\frac{t}{2}]$ we have

$$\begin{aligned}
&\int_0^{\frac{t}{2}} b(\tau)^{-1}(1+B(t,\tau))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s}{2}}\||u(\tau,\cdot)|^q\|_{L^m(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)}d\tau\\
&\lesssim \|(u,v)\|^q_{X(t)}\int_0^{\frac{t}{2}} b(\tau)^{-1}(1+B(t,\tau))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s}{2}}(1+B(\tau,0))^{-\frac{n}{2m}q+\frac{n}{2m}+\gamma_{n,m}(p)q}d\tau\\
&\lesssim \|(u,v)\|^q_{X(t)}(1+B(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s}{2}}\int_0^{\frac{t}{2}} b(\tau)^{-1}(1+B(\tau,0))^{-\frac{n}{2m}q+\frac{n}{2m}+\gamma_{n,m}(p)q}d\tau\\
&\lesssim (1+B(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s}{2}}\|(u,v)\|^q_{X(t)},
\end{aligned}$$

where we take account of $-\frac{n}{2m}q+\frac{n}{2m}+\gamma_{n,m}(p)q<-1$ which is equivalent to (2.10).
For $\tau\in[\frac{t}{2},t]$ we have

$$\begin{aligned}
&\int_{\frac{t}{2}}^{t} b(\tau)^{-1}(1+B(t,\tau))^{-\frac{s}{2}}\||u(\tau,x)|^q\|_{L^2(\mathbb{R}^n)}d\tau\\
&\lesssim \|(u,v)\|^q_{X(t)}\int_{\frac{t}{2}}^{t} b(\tau)^{-1}(1+B(t,\tau))^{-\frac{s}{2}}(1+B(\tau,0))^{-\frac{n}{2m}q+\frac{n}{4}+\gamma_{n,m}(p)q}d\tau\\
&\lesssim (1+B(t,0))^{1-\frac{s}{2}-\frac{n}{2m}q+\frac{n}{4}+\gamma_{n,m}(p)q}\|(u,v)\|^q_{X(t)}\\
&\lesssim (1+B(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s}{2}}\|(u,v)\|^q_{X(t)}
\end{aligned}$$

where the assumption (2.10) is used again.
Finally, we get

$$\big\||D|^s v^{nl}(t,\cdot)\big\|_{L^2(\mathbb{R}^n)} \lesssim (1+B(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s}{2}}\|(u,v)\|^q_{X(t)}. \tag{2.17}$$

Analogously, we obtain

$$\big\|v^{nl}(t,\cdot)\big\|_{L^2(\mathbb{R}^n)} \lesssim (1+B(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)}\|(u,v)\|^q_{X(t)}. \tag{2.18}$$

From (2.15) to (2.18) we get

$$\|(u^{nl},v^{nl})\|_{X(t)} \lesssim \|(u,v)\|^p_{X(t)}+\|(u,v)\|^q_{X(t)}. \tag{2.19}$$

Summarizing, (2.12) can be concluded from (2.14) and (2.19).
To prove (2.13) we assume that (u,v) and $(\tilde{u},\tilde{v})$ are two vector-functions belonging

to $X(t)$. Then, we have

$$\begin{aligned}
&N(u,v)-N(\tilde{u},\tilde{v})\\
&\quad=\big(u^{nl}(t,x)-\tilde{u}^{nl}(t,x),v^{nl}(t,x)-\tilde{v}^{nl}(t,x)\big)\\
&\quad=\Big(\int_0^t E_1(t,\tau,x)*_{(x)}\big(|v(\tau,x)|^p-|\tilde{v}(\tau,x)|^p\big)d\tau,\\
&\qquad\quad\int_0^t E_1(t,\tau,x)*_{(x)}\big(|u(\tau,x)|^q-|\tilde{u}(\tau,x)|^q\big)d\tau\Big).
\end{aligned}$$

To estimate the first component we use the estimate (1.25) from Theorem 1.5, with $m=2$ if $\tau\in[\frac{t}{2},t]$. In this way we get

$$\begin{aligned}
&\Big\||D|^s\int_0^t E_1(t,\tau,x)*_{(x)}\big(|v(\tau,x)|^p-|\tilde{v}(\tau,x)|^p\big)d\tau\Big\|_{L^2(\mathbb{R}^n)}\\
&\lesssim\int_0^{\frac{t}{2}}b(\tau)^{-1}(1+B(t,\tau))^{-\frac{n}{2}(\frac{1}{m}-\frac{1}{2})-\frac{s}{2}}\||v(\tau,x)|^p-|\tilde{v}(\tau,x)|^p\|_{L^m(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)}d\tau\\
&\quad+\int_{\frac{t}{2}}^t b(\tau)^{-1}(1+B(t,\tau))^{-\frac{s}{2}}\||v(\tau,x)|^p-|\tilde{v}(\tau,x)|^p\|_{L^2(\mathbb{R}^n)}d\tau.
\end{aligned}$$

Using Hölder's inequality we obtain the following inequalities:

$$\big\||v(\tau,x)|^p-|\tilde{v}(\tau,x)|^p\big\|_{L^m(\mathbb{R}^n)}\lesssim\big\|v(\tau,\cdot)-\tilde{v}(\tau,\cdot)\big\|_{L^{mp}(\mathbb{R}^n)}\big(\|v(\tau,\cdot)\|^{p-1}_{L^{mp}(\mathbb{R}^n)}+\|\tilde{v}(\tau,\cdot)\|^{p-1}_{L^{mp}(\mathbb{R}^n)}\big),$$

$$\big\||v(\tau,x)|^p-|\tilde{v}(\tau,x)|^p\big\|_{L^2(\mathbb{R}^n)}\lesssim\big\|v(\tau,\cdot)-\tilde{v}(\tau,\cdot)\big\|_{L^{2p}(\mathbb{R}^n)}\big(\|v(\tau,\cdot)\|^{p-1}_{L^{2p}(\mathbb{R}^n)}+\|\tilde{v}(\tau,\cdot)\|^{p-1}_{L^{2p}(\mathbb{R}^n)}\big).$$

By using the Gagliardo-Nirenberg inequality we can derive, as we did in the existence part of the proof, for $0\le\tau\le t$ the following estimates:

$$\begin{aligned}
\big\|v(\tau,\cdot)-\tilde{v}(\tau,\cdot)\big\|_{L^{mp}(\mathbb{R}^n)}&\lesssim(1+B(\tau,0))^{-\frac{n}{2m}+\frac{n}{2mp}}\sup_{\tau\in[0,t]}M(\tau,v-\tilde{v}),\\
\|v(\tau,\cdot)\|^{p-1}_{L^{mp}(\mathbb{R}^n)}&\lesssim(1+B(\tau,0))^{-\frac{n}{2m}(p-1)+\frac{n}{2mp}(p-1)}\|(u,v)\|^{p-1}_{X(t)},\\
\|\tilde{v}(\tau,\cdot)\|^{p-1}_{L^{mp}(\mathbb{R}^n)}&\lesssim(1+B(\tau,0))^{-\frac{n}{2m}(p-1)+\frac{n}{2mp}(p-1)}\|(\tilde{u},\tilde{v})\|^{p-1}_{X(t)},\\
\big\|v(\tau,\cdot)-\tilde{v}(\tau,\cdot)\big\|_{L^{2p}(\mathbb{R}^n)}&\lesssim(1+B(\tau,0))^{-\frac{n}{2m}+\frac{n}{4p}}\sup_{\tau\in[0,t]}M(\tau,v-\tilde{v}),\\
\|v(\tau,\cdot)\|^{p-1}_{L^{2p}(\mathbb{R}^n)}&\lesssim(1+B(\tau,0))^{-\frac{n}{2m}(p-1)+\frac{n}{4p}(p-1)}\|(u,v)\|^{p-1}_{X(t)},\\
\|\tilde{v}(\tau,\cdot)\|^{p-1}_{L^{2p}(\mathbb{R}^n)}&\lesssim(1+B(\tau,0))^{-\frac{n}{2m}(p-1)+\frac{n}{4p}(p-1)}\|(\tilde{u},\tilde{v})\|^{p-1}_{X(t)}.
\end{aligned}$$

Using the last estimates and following the same way to prove (2.15) and (2.16) we can derive

$$\begin{aligned}
&\Big\||D|^s\textstyle\int_0^t E_1(t,\tau,x)*_{(x)}\big(|v(\tau,x)|^p-|\tilde{v}(\tau,x)|^p\big)d\tau\Big\|_{L^2(\mathbb{R}^n)}\\
&\qquad\lesssim(1+B(t,0))^{-\frac{n}{2}(\frac{1}{m}-\frac{1}{2})-\frac{s}{2}+\gamma_{n,m}(p)}\sup_{\tau\in[0,t]}M(\tau,v-\tilde{v})\\
&\qquad\quad\times\Big(\|(u,v)\|^{p-1}_{X(t)}+\|(\tilde{u},\tilde{v})\|^{p-1}_{X(t)}\Big),
\end{aligned}\tag{2.20}$$

and

$$\begin{aligned}\Big\|\int_0^t E_1(t,\tau,x) *_{(x)} \big(|v(\tau,x)|^p - |\tilde{v}(\tau,x)|^p\big)d\tau\Big\|_{L^2(\mathbb{R}^n)} \\ \lesssim (1+B(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)+\gamma_{n,m}(p)} \sup_{\tau\in[0,t]} M(\tau, v-\tilde{v}) \\ \times \Big(\|(u,v)\|^{p-1}_{X(t)} + \|(\tilde{u},\tilde{v})\|^{p-1}_{X(t)}\Big).\end{aligned} \tag{2.21}$$

Analogously, we have for $0 \le \tau \le t$ the following estimates:

$$\begin{aligned}\big\|u(\tau,\cdot)-\tilde{u}(\tau,\cdot)\big\|_{L^{mq}(\mathbb{R}^n)} &\lesssim (1+B(\tau,0))^{-\frac{n}{2m}+\frac{n}{2mq}+\gamma_{n,m}(p)} \sup_{\tau\in[0,t]} M(\tau,u-\tilde{u}),\\ \|u(\tau,\cdot)\|^{q-1}_{L^{mq}(\mathbb{R}^n)} &\lesssim (1+B(\tau,0))^{\left(-\frac{n}{2m}+\frac{n}{2mq}+\gamma_{n,m}(p)\right)(q-1)}\|(u,v)\|^{q-1}_{X(t)},\\ \|\tilde{u}(\tau,\cdot)\|^{q-1}_{L^{mq}(\mathbb{R}^n)} &\lesssim (1+B(\tau,0))^{\left(-\frac{n}{2m}+\frac{n}{2mq}+\gamma_{n,m}(p)\right)(q-1)}\|(\tilde{u},\tilde{v})\|^{q-1}_{X(t)},\\ \big\|u(\tau,\cdot)-\tilde{u}(\tau,\cdot)\big\|_{L^{2q}(\mathbb{R}^n)} &\lesssim (1+B(\tau,0))^{-\frac{n}{2m}+\frac{n}{2q}+\gamma_{n,2}(p)} \sup_{\tau\in[0,t]} M(\tau,u-\tilde{u}),\\ \|u(\tau,\cdot)\|^{q-1}_{L^{2q}(\mathbb{R}^n)} &\lesssim (1+B(\tau,0))^{\left(-\frac{n}{2m}+\frac{n}{2q}+\gamma_{n,2}(p)\right)(q-1)}\|(u,v)\|^{q-1}_{X(t)},\\ \|\tilde{u}(\tau,\cdot)\|^{q-1}_{L^{2q}(\mathbb{R}^n)} &\lesssim (1+B(\tau,0))^{\left(-\frac{n}{2m}+\frac{n}{2q}+\gamma_{n,2}(p)\right)(q-1)}\|(\tilde{u},\tilde{v})\|^{q-1}_{X(t)}.\end{aligned}$$

Provided that (2.10) is satisfied these estimates lead to

$$\begin{aligned}\Big\||D|^s \int_0^t E_1(t,\tau,x) *_{(x)} \big(|u(\tau,x)|^q - |\tilde{u}(\tau,x)|^q\big)ds\Big\|_{L^2(\mathbb{R}^n)} \\ \lesssim (1+B(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s}{2}} \sup_{\tau\in[0,t]} M(\tau,u-\tilde{u}) \\ \times \Big(\|(u,v)\|^{q-1}_{X(t)} + \|(\tilde{u},\tilde{v})\|^{q-1}_{X(t)}\Big),\end{aligned} \tag{2.22}$$

and

$$\begin{aligned}\Big\|\int_0^t E_1(t,\tau,x) *_{(x)} \big(|u(\tau,x)|^q - |\tilde{u}(\tau,x)|^q\big)d\tau\Big\|_{L^2(\mathbb{R}^n)} \\ \lesssim (1+B(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)} \sup_{\tau\in[0,t]} M(\tau,u-\tilde{u}) \\ \times \Big(\|(u,v)\|^{q-1}_{X(t)} + \|(\tilde{u},\tilde{v})\|^{q-1}_{X(t)}\Big).\end{aligned} \tag{2.23}$$

To complete the proof it is sufficient to estimate $\|N(u,v)-N(\tilde{u},\tilde{v})\|_{X(t)}$ after using (2.20) to (2.23). □

If we are interested in the case $p > q$, then we get an existence and uniqueness result with a loss of decay appearing in v.

Theorem 2.4. *Let $n < \min\{\frac{2m^2}{2-m};\frac{2sm}{m-s}\}$. The data (u_0,u_1) and (v_0,v_1) are assumed to belong to $\mathcal{A}_{m,s}\times\mathcal{A}_{m,s}$, $s\in(0,1)$ and $m\in[1,2)$. Moreover, let*

$$\frac{2}{m} \le q < p_{Fuj,m}(n) < p \le p_{GN,s}(n)$$

and

$$\frac{n}{2} > m\Big(\frac{p+1}{pq-1}\Big). \tag{2.24}$$

Then, there exists a small constant ϵ_0 such that if

$$\|(u_0,u_1)\|_{\mathcal{A}_{m,s}} + \|(v_0,v_1)\|_{\mathcal{A}_{m,s}} \leq \epsilon_0,$$

then there exists a uniquely determined globally (in time) Sobolev solution to (2.1) in

$$\left(\mathcal{C}([0,\infty), H^s(\mathbb{R}^n))\right)^2.$$

Furthermore, the solution satisfies the following estimates:

$$\begin{aligned}
\|u(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \big(1+B(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)} \left(\|(u_0,u_1)\|_{\mathcal{A}_{m,s}} + \|(v_0,v_1)\|_{\mathcal{A}_{m,s}}\right),\\
\||D|^s u(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \big(1+B(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s}{2}} \left(\|(u_0,u_1)\|_{\mathcal{A}_{m,s}} + \|(v_0,v_1)\|_{\mathcal{A}_{m,s}}\right),\\
\|v(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \big(1+B(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)+\gamma_{n,m}(q)} \left(\|(u_0,u_1)\|_{\mathcal{A}_{m,s}} + \|(v_0,v_1)\|_{\mathcal{A}_{m,s}}\right),\\
\||D|^s v(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \big(1+B(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s}{2}+\gamma_{n,m}(q)} \left(\|(u_0,u_1)\|_{\mathcal{A}_{m,s}} + \|(v_0,v_1)\|_{\mathcal{A}_{m,s}}\right),
\end{aligned}$$

where $\gamma_{n,m}(q) = -\frac{n}{2m}(q-1)+1$ represents the loss of decay of solutions in comparison with the corresponding decay estimates for the solution v to the linear Cauchy problem with vanishing right-hand side.

From Theorems 2.3 and 2.4 we may conclude the following corollary.

Corollary 2.1. *Let $n < \min\{\frac{2m^2}{2-m}; \frac{2sm}{m-s}\}$. The data $(u_0,u_1),(v_0,v_1)$ are assumed to belong to $\mathcal{A}_{m,s}\times\mathcal{A}_{m,s}$, $s\in(0,1)$ and $m\in[1,2)$. Moreover, let*

$$\frac{2}{m} \leq \min\{p;q\} < p_{Fuj,m}(n) < \max\{p;q\} \leq p_{GN,s}(n) \tag{2.25}$$

and

$$\frac{n}{2} > m\Big(\frac{\max\{p;q\}+1}{pq-1}\Big). \tag{2.26}$$

Then, there exists a small constant ϵ_0 such that if

$$\|(u_0,u_1)\|_{\mathcal{A}_{m,s}} + \|(v_0,v_1)\|_{\mathcal{A}_{m,s}} \leq \epsilon_0,$$

then there exists a uniquely determined globally (in time) Sobolev solution to (2.1) in

$$\left(\mathcal{C}([0,\infty), H^s(\mathbb{R}^n))\right)^2.$$

Furthermore, the solution satisfies the decay estimates:

$$\begin{aligned}
\|u(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \big(1+B(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)+[\gamma_{n,m}(p)]^+} \left(\|(u_0,u_1)\|_{\mathcal{A}_{m,s}} + \|(v_0,v_1)\|_{\mathcal{A}_{m,s}}\right),\\
\||D|^s u(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \big(1+B(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s}{2}+[\gamma_{n,m}(p)]^+} \left(\|(u_0,u_1)\|_{\mathcal{A}_{m,s}} + \|(v_0,v_1)\|_{\mathcal{A}_{m,s}}\right),\\
\|v(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \big(1+B(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)+[\gamma_{n,m}(q)]^+} \left(\|(u_0,u_1)\|_{\mathcal{A}_{m,s}} + \|(v_0,v_1)\|_{\mathcal{A}_{m,s}}\right),\\
\||D|^s v(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \big(1+B(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s}{2}+[\gamma_{n,m}(q)]^+} \left(\|(u_0,u_1)\|_{\mathcal{A}_{m,s}} + \|(v_0,v_1)\|_{\mathcal{A}_{m,s}}\right),
\end{aligned}$$

where $[\gamma_{n,m}(p)]^+ = \max\{\gamma_{n,m}(p);0\}$ and $[\gamma_{n,m}(q)]^+ = \max\{\gamma_{n,m}(q);0\}$.

2.1.3. Different additional regularities

In this section we present results in the case, where the data have different additional regularities, namely

$$\big((u_0, u_1), (v_0, v_1)\big) \in \mathcal{A}_{m_1,s} \times \mathcal{A}_{m_2,s},$$

where $s \in (0,1)$ and $m_1, m_2 \in [1,2]$. We will prove a global (in time) existence result for small data Sobolev solutions. The pivotal condition for power nonlinearities p and q are different and even at most one exponent can be smaller than the modified Fujita exponent without any loss of decay which is impossible in the case $m_1 = m_2$.

Theorem 2.5. *Let $n \leq 2$ and the data (u_0, u_1) and (v_0, v_1) are assumed to belong to $\mathcal{A}_{m_1,s} \times \mathcal{A}_{m_2,s}$, $s \in (0,1)$ and $m_1, m_2 \in [1,2]$. Moreover, we suppose the following conditions for the exponents p and q:*

$$p > \frac{m_2}{m_1} + \frac{2m_2}{n}, \qquad \frac{2}{m_1} \leq p \leq p_{GN,s}(n), \tag{2.27}$$

$$q > \frac{m_1}{m_2} + \frac{2m_1}{n}, \qquad \frac{2}{m_2} \leq q \leq p_{GN,s}(n). \tag{2.28}$$

Then, there exists a small constant ϵ_0 such that if

$$\|(u_0, u_1)\|_{\mathcal{A}_{m_1,s}} + \|(v_0, v_1)\|_{\mathcal{A}_{m_2,s}} \leq \epsilon_0,$$

then there exists a uniquely determined globally (in time) Sobolev solution to (2.1) in

$$\left(\mathcal{C}([0,\infty), H^s(\mathbb{R}^n))\right)^2.$$

Furthermore, the solution satisfies the following decay estimates:

$$\begin{aligned}
\|u(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \big(1+B(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m_1}-\frac{1}{2}\right)} \left(\|(u_0,u_1)\|_{\mathcal{A}_{m_1,s}} + \|(v_0,v_1)\|_{\mathcal{A}_{m_2,s}}\right), \\
\|v(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \big(1+B(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m_2}-\frac{1}{2}\right)} \left(\|(u_0,u_1)\|_{\mathcal{A}_{m_1,s}} + \|(v_0,v_1)\|_{\mathcal{A}_{m_2,s}}\right), \\
\big\||D|^s u(t,\cdot)\big\|_{L^2(\mathbb{R}^n)} &\lesssim \big(1+B(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m_1}-\frac{1}{2}\right)-\frac{s}{2}} \left(\|(u_0,u_1)\|_{\mathcal{A}_{m_1,s}} + \|(v_0,v_1)\|_{\mathcal{A}_{m_2,s}}\right), \\
\big\||D|^s v(t,\cdot)\big\|_{L^2(\mathbb{R}^n)} &\lesssim \big(1+B(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m_2}-\frac{1}{2}\right)-\frac{s}{2}} \left(\|(u_0,u_1)\|_{\mathcal{A}_{m_1,s}} + \|(v_0,v_1)\|_{\mathcal{A}_{m_2,s}}\right).
\end{aligned}$$

Remark 2.5. *We remark that in the case $p \leq q$ it is better to choose $m_1 > m_2$ which implies a lower bound for p smaller than that one for q. This leads simultaneously, when the exponent q is sufficiently large, to a global (in time) existence result for p smaller than the modified Fujita exponent $p_{Fuj,m_2}(n)$ without any loss of decay. This effect does not appear in the previous cases where $m_1 = m_2$. The following example illustrates this effect.*

Example 2.2. *For $n = 1$, $s = \frac{1}{2}$, $m_1 = 2$ and $m_2 = 1$ we obtain a global (in time) existence result for $q > 2 + \frac{4}{n} = 6$ and $p > \frac{1}{2} + \frac{2}{n} = \frac{5}{2}$. Here the lower bound for p is smaller than the modified Fujita exponent $p_{Fuj,1}(1) = 3$.*

Remark 2.6. *In Theorem 2.5 we restrict ourselves to the case $n \leq 2$ because the conditions on the exponents p and q are more complicate for $n \geq 3$ and influenced by several parameters m_1, m_2 and the regularity parameter of the data s. The following example clarifies the situation in a particular case for $n = 3$ and shows how the different additional regularities restrict the choice of s and the admissible range for p and q.*

Example 2.3. *Let $n = 3$, $m_1 = \frac{7}{4}$ and $m_2 = \frac{5}{4}$. Then to get a non-empty admissible range for the exponent q we have to take the regularity parameter for the data s close to 1. Choosing $s = 0.95$ leads to a global (in time) existence result for*

$$\frac{65}{42} < p \leq p_{GN,0.95}(3), \qquad \frac{77}{30} < q \leq p_{GN,0.95}(3).$$

As a remark, the lower bound for p is smaller than $p_{Fuj,m_2}(3)$.

Proof. Let us define the solution space $X(t)$ by

$$X(t) = \big\{(u,v) \in \big(\mathcal{C}([0,t], H^s(\mathbb{R}^n))\big)^2\big\},$$

with the norm

$$\|(u,v)\|_{X(t)} = \sup_{\tau\in[0,t]} \big\{M_1(\tau,u) + M_2(\tau,v)\big\},$$

where

$$\begin{aligned}
M_1(\tau,u) &= \big(1+B(\tau,0)\big)^{\frac{n}{2}\left(\frac{1}{m_1}-\frac{1}{2}\right)}\|u(\tau,\cdot)\|_{L^2(\mathbb{R}^n)} \\
&\quad +\big(1+B(\tau,0)\big)^{\frac{n}{2}\left(\frac{1}{m_1}-\frac{1}{2}\right)+\frac{s}{2}}\||D|^s u(\tau,\cdot)\|_{L^2(\mathbb{R}^n)}, \\
M_2(\tau,v) &= \big(1+B(\tau,0)\big)^{\frac{n}{2}\left(\frac{1}{m_2}-\frac{1}{2}\right)}\|v(\tau,\cdot)\|_{L^2(\mathbb{R}^n)} \\
&\quad +\big(1+B(\tau,0)\big)^{\frac{n}{2}\left(\frac{1}{m_2}-\frac{1}{2}\right)+\frac{s}{2}}\||D|^s v(\tau,\cdot)\|_{L^2(\mathbb{R}^n)}.
\end{aligned}$$

Let N be the mapping as defined in (2.11) by

$$N\,:\,(u,v)\in X(t) \to N(u,v) = (u^{ln}+u^{nl}, v^{ln}+v^{nl}).$$

As we did in the proofs of the previous theorems of this section, our goal is to prove the following estimates:

$$\|N(u,v)\|_{X(t)} \lesssim \|(u_0,u_1)\|_{\mathcal{A}_{m_1,s}} + \|(v_0,v_1)\|_{\mathcal{A}_{m_2,s}} + \|(u,v)\|^p_{X(t)} + \|(u,v)\|^q_{X(t)}, \tag{2.29}$$

$$\begin{aligned}
\|N(u,v)-N(\tilde{u},\tilde{v})\|_{X(t)} &\lesssim \|(u,v)-(\tilde{u},\tilde{v})\|_{X(t)} \\
&\times\big(\|(u,v)\|^{p-1}_{X(t)} + \|(\tilde{u},\tilde{v})\|^{p-1}_{X(t)} + \|(u,v)\|^{q-1}_{X(t)} + \|(\tilde{u},\tilde{v})\|^{q-1}_{X(t)}\big).
\end{aligned} \tag{2.30}$$

From the definition of the norm of solution space $X(t)$ and the estimates of Theorem 1.2 one can immediately conclude

$$\|(u^{ln},v^{ln})\|_{X(t)} \lesssim \|(u_0,u_1)\|_{\mathcal{A}_{m_1,s}} + \|(v_0,v_1)\|_{\mathcal{A}_{m_2,s}}. \tag{2.31}$$

To prove the first inequality it remains to estimate the nonlinear terms. Let us begin with u^{nl}. Using the estimate (1.25) from Theorem 1.5, with $m_1 = 2$ for the integral over $[\frac{t}{2}, t]$, we get

$$\begin{aligned}
&\big\||D|^s u^{nl}(t,\cdot)\big\|_{L^2(\mathbb{R}^n)} \\
&\quad\lesssim \int_0^{\frac{t}{2}} b(\tau)^{-1}(1+B(t,\tau))^{-\frac{n}{2}\left(\frac{1}{m_1}-\frac{1}{2}\right)-\frac{s}{2}} \||v(\tau,x)|^p\|_{L^{m_1}(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)} d\tau \\
&\qquad + \int_{\frac{t}{2}}^t b(\tau)^{-1}(1+B(t,\tau))^{-\frac{s}{2}} \||v(\tau,x)|^p\|_{L^2(\mathbb{R}^n)} d\tau. \qquad (2.32)
\end{aligned}$$

Similar to (1.34) and (1.35) we obtain for $0 \le \tau \le t$ the following estimates:

$$\||v(\tau,x)|^p\|_{L^{m_1}(\mathbb{R}^n)} \lesssim (1+B(\tau,0))^{-\frac{n}{2m_2}p+\frac{n}{2m_1}} \|(u,v)\|^p_{X(t)}, \qquad (2.33)$$

and

$$\||v(\tau,x)|^p\|_{L^2(\mathbb{R}^n)} \lesssim (1+B(\tau,0))^{-\frac{n}{2m_2}p+\frac{n}{4}} \|(u,v)\|^p_{X(t)}, \qquad (2.34)$$

where $\frac{2}{m_1} \le p \le p_{GN,s}(n)$ due to the use of the Gagliardo-Nirenberg inequality. Plugging the last estimates into the estimates of the norm $\big\||D|^s u^{nl}(t,\cdot)\big\|_{L^2(\mathbb{R}^n)}$ and taking into consideration (1.6), (1.7) we obtain

$$\begin{aligned}
&\big\||D|^s u^{nl}(t,\cdot)\big\|_{L^2(\mathbb{R}^n)} \\
&\lesssim \|(u,v)\|^p_{X(t)} \int_0^{\frac{t}{2}} b(\tau)^{-1}(1+B(t,\tau))^{-\frac{n}{2}\left(\frac{1}{m_1}-\frac{1}{2}\right)-\frac{s}{2}}(1+B(\tau,0))^{-\frac{n}{2m_2}p+\frac{n}{2m_1}} d\tau \\
&\quad + \|(u,v)\|^p_{X(t)} \int_{\frac{t}{2}}^t b(\tau)^{-1}(1+B(t,\tau))^{-\frac{s}{2}}(1+B(\tau,0))^{-\frac{n}{2m_2}p+\frac{n}{4}} d\tau \\
&\lesssim \|(u,v)\|^p_{X(t)}(1+B(t,0))^{-\frac{n}{2}\left(\frac{1}{m_1}-\frac{1}{2}\right)-\frac{s}{2}} \int_0^{\frac{t}{2}} b(\tau)^{-1}(1+B(\tau,0))^{-\frac{n}{2m_2}p+\frac{n}{2m_1}} d\tau \\
&\quad + \|(u,v)\|^p_{X(t)}(1+B(t,0))^{-\frac{n}{2m_2}p+\frac{n}{4}+1-\frac{s}{2}} \\
&\lesssim \|(u,v)\|^p_{X(t)}(1+B(t,0))^{-\frac{n}{2}\left(\frac{1}{m_1}-\frac{1}{2}\right)-\frac{s}{2}},
\end{aligned}$$

where we use $p > \frac{m_2}{m_1} + \frac{2m_2}{n}$ which is included in (2.27). Then,

$$\big\||D|^s u^{nl}(t,\cdot)\big\|_{L^2(\mathbb{R}^n)} \lesssim \|(u,v)\|^p_{X(t)}(1+B(t,0))^{-\frac{n}{2}\left(\frac{1}{m_1}-\frac{1}{2}\right)-\frac{s}{2}}. \qquad (2.35)$$

In the same way we can prove

$$\big\|u^{nl}(t,\cdot)\big\|_{L^2(\mathbb{R}^n)} \lesssim \|(u,v)\|^p_{X(t)}(1+B(t,0))^{-\frac{n}{2}\left(\frac{1}{m_1}-\frac{1}{2}\right)}. \qquad (2.36)$$

Now, for the second component v^{nl} we follow the same steps to derive the estimate for u^{nl} after changing the roles of m_1 by m_2. Supposing that (2.28) is satisfied this leads to the following estimates:

$$\big\||D|^s v^{nl}(t,\cdot)\big\|_{L^2(\mathbb{R}^n)} \lesssim \|(u,v)\|^q_{X(t)}(1+B(t,0))^{-\frac{n}{2}\left(\frac{1}{m_2}-\frac{1}{2}\right)-\frac{s}{2}}, \qquad (2.37)$$

$$\left\|v^{nl}(t,\cdot)\right\|_{L^2(\mathbb{R}^n)} \lesssim \|(u,v)\|_{X(t)}^q (1+B(t,0))^{-\frac{n}{2}\left(\frac{1}{m_2}-\frac{1}{2}\right)}. \tag{2.38}$$

Finally, (2.35) to (2.38) with (2.31) implies (2.29).
To prove (2.30) we assume that (u,v) and $(\tilde{u},\tilde{v})$ are two vector-functions belonging to $X(t)$. Then we have

$$\begin{aligned}
&N(u,v)-N(\tilde{u},\tilde{v})\\
&\quad=(u^{nl}(t,x)-\tilde{u}^{nl}(t,x), v^{nl}(t,x)-\tilde{v}^{nl}(t,x))\\
&\quad=\Big(\int_0^t E_1(t,\tau,x)*_{(x)}\big(|v(\tau,x)|^p-|\tilde{v}(\tau,x)|^p\big)d\tau,\\
&\qquad\quad \int_0^t E_1(t,\tau,x)*_{(x)}\big(|u(\tau,x)|^q-|\tilde{u}(\tau,x)|^q\big)d\tau\Big).
\end{aligned}$$

Using the estimate (1.25) from Theorem 1.5, with $m_1=2$ or $m_2=2$ for $\tau\in[\frac{t}{2},t]$, we get

$$\begin{aligned}
&\Big\||D|^s\int_0^t E_1(t,\tau,x)*_{(x)}\big(|v(\tau,x)|^p-|\tilde{v}(\tau,x)|^p\big)d\tau\Big\|_{L^2(\mathbb{R}^n)}\\
&\lesssim \int_0^{\frac{t}{2}} b(\tau)^{-1}(1+B(t,\tau))^{-\frac{n}{2}\left(\frac{1}{m_1}-\frac{1}{2}\right)-\frac{s}{2}}\||v(\tau,x)|^p-|\tilde{v}(\tau,x)|^p\|_{L^{m_1}(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)}d\tau\\
&\quad+\int_{\frac{t}{2}}^t b(\tau)^{-1}(1+B(t,\tau))^{-\frac{s}{2}}\||v(\tau,x)|^p-|\tilde{v}(\tau,x)|^p\|_{L^2(\mathbb{R}^n)}d\tau,
\end{aligned}$$

and

$$\begin{aligned}
&\Big\||D|^s\int_0^t E_1(t,\tau,x)*_{(x)}\big(|u(\tau,x)|^q-|\tilde{u}(\tau,x)|^q\big)d\tau\Big\||_{L^2(\mathbb{R}^n)}\\
&\lesssim \int_0^{\frac{t}{2}} b(\tau)^{-1}(1+B(t,\tau))^{-\frac{n}{2}\left(\frac{1}{m_2}-\frac{1}{2}\right)-\frac{s}{2}}\||u(\tau,x)|^q-|\tilde{u}(\tau,x)|^q\|_{L^{m_2}(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)}d\tau\\
&\quad+\int_{\frac{t}{2}}^t b(\tau)^{-1}(1+B(t,\tau))^{-\frac{s}{2}}\||u(\tau,x)|^q-|\tilde{u}(\tau,x)|^q\|_{L^2(\mathbb{R}^n)}d\tau.
\end{aligned}$$

Using Hölder's inequality we obtain

$$\begin{aligned}
&\big\||v(\tau,x)|^p-|\tilde{v}(\tau,x)|^p\big\|_{L^{m_1}(\mathbb{R}^n)}\\
&\qquad\lesssim \big\|v(\tau,\cdot)-\tilde{v}(\tau,\cdot)\big\|_{L^{m_1p}(\mathbb{R}^n)}\big(\|v(\tau,\cdot)\|^{p-1}_{L^{m_1p}(\mathbb{R}^n)}+\|\tilde{v}(\tau,\cdot)\|^{p-1}_{L^{m_1p}(\mathbb{R}^n)}\big),\\
&\big\||v(\tau,x)|^p-|\tilde{v}(\tau,x)|^p\big\|_{L^2(\mathbb{R}^n)}\\
&\qquad\lesssim \big\|v(\tau,\cdot)-\tilde{v}(\tau,\cdot)\big\|_{L^{2p}(\mathbb{R}^n)}\big(\|v(\tau,\cdot)\|^{p-1}_{L^{2p}(\mathbb{R}^n)}+\|\tilde{v}(\tau,\cdot)\|^{p-1}_{L^{2p}(\mathbb{R}^n)}\big),\\
&\big\||u(\tau,\cdot)|^q-|\tilde{u}(\tau,\cdot)|^q\big\|_{L^{m_2}(\mathbb{R}^n)}\\
&\qquad\lesssim \big\|u(\tau,\cdot)-\tilde{u}(\tau,\cdot)\big\|_{L^{m_2q}(\mathbb{R}^n)}\big(\|u(\tau,\cdot)\|^{q-1}_{L^{m_2q}(\mathbb{R}^n)}+\|\tilde{u}(\tau,\cdot)\|^{q-1}_{L^{m_2q}(\mathbb{R}^n)}\big),\\
&\big\||u(\tau,\cdot)|^q-|\tilde{u}(\tau,\cdot)|^q\big\|_{L^2(\mathbb{R}^n)}\\
&\qquad\lesssim \big\|u(\tau,\cdot)-\tilde{u}(\tau,\cdot)\big\|_{L^{2q}(\mathbb{R}^n)}\big(\|u(\tau,\cdot)\|^{q-1}_{L^{2q}(\mathbb{R}^n)}+\|\tilde{u}(\tau,\cdot)\|^{q-1}_{L^{2q}(\mathbb{R}^n)}\big).
\end{aligned}$$

Using Gagliardo-Nirenberg inequality one can prove for $0 \leq \tau \leq t$ the following estimates:

$$\begin{aligned}
\big\|v(\tau,\cdot)-\tilde{v}(\tau,\cdot)\big\|_{L^{m_1p}(\mathbb{R}^n)} &\lesssim (1+B(\tau,0))^{-\frac{n}{2m_2}+\frac{n}{2m_1p}} \sup_{\tau\in[0,t]} M(\tau,v-\tilde{v}),\\
\|v(\tau,\cdot)\|^{p-1}_{L^{m_1p}(\mathbb{R}^n)} &\lesssim (1+B(\tau,0))^{-\frac{n}{2m_2}(p-1)+\frac{n}{2m_1p}(p-1)}\|(u,v)\|^{p-1}_{X(t)},\\
\|\tilde{v}(\tau,\cdot)\|^{p-1}_{L^{m_1p}(\mathbb{R}^n)} &\lesssim (1+B(\tau,0))^{-\frac{n}{2m_2}(p-1)+\frac{n}{2m_1p}(p-1)}\|(\tilde{u},\tilde{v})\|^{p-1}_{X(t)},\\
\big\|v(\tau,\cdot)-\tilde{v}(\tau,\cdot)\big\|_{L^{2p}(\mathbb{R}^n)} &\lesssim (1+B(\tau,0))^{-\frac{n}{2m_2}+\frac{n}{4p}} \sup_{\tau\in[0,t]} M(\tau,v-\tilde{v}),\\
\|v(\tau,\cdot)\|^{p-1}_{L^{2p}(\mathbb{R}^n)} &\lesssim (1+B(\tau,0))^{-\frac{n}{2m_2}(p-1)+\frac{n}{4p}(p-1)}\|(u,v)\|^{p-1}_{X(t)},\\
\|\tilde{v}(\tau,\cdot)\|^{p-1}_{L^{2p}(\mathbb{R}^n)} &\lesssim (1+B(\tau,0))^{-\frac{n}{2m_2}(p-1)+\frac{n}{4p}(p-1)}\|(\tilde{u},\tilde{v})\|^{p-1}_{X(t)},\\
\big\|u(\tau,\cdot)-\tilde{u}(\tau,\cdot)\big\|_{L^{m_2q}(\mathbb{R}^n)} &\lesssim (1+B(\tau,0))^{-\frac{n}{2m_1}+\frac{n}{2m_2q}} \sup_{\tau\in[0,t]} M(\tau,u-\tilde{u}),\\
\|u(\tau,\cdot)\|^{q-1}_{L^{m_2q}(\mathbb{R}^n)} &\lesssim (1+B(\tau,0))^{\left(-\frac{n}{2m_1}+\frac{n}{2m_2q}\right)(q-1)}\|(u,v)\|^{q-1}_{X(t)},\\
\|\tilde{u}(\tau,\cdot)\|^{q-1}_{L^{m_2q}(\mathbb{R}^n)} &\lesssim (1+B(\tau,0))^{\left(-\frac{n}{2m_1}+\frac{n}{2m_2q}\right)(q-1)}\|(\tilde{u},\tilde{v})\|^{q-1}_{X(t)},\\
\big\|u(\tau,\cdot)-\tilde{u}(\tau,\cdot)\big\|_{L^{2q}(\mathbb{R}^n)} &\lesssim (1+B(\tau,0))^{-\frac{n}{2m_1}+\frac{n}{4q}} \sup_{\tau\in[0,t]} M(\tau,u-\tilde{u}),\\
\|u(\tau,\cdot)\|^{q-1}_{L^{2q}(\mathbb{R}^n)} &\lesssim (1+B(\tau,0))^{\left(-\frac{n}{2m_1}+\frac{n}{4q}\right)(q-1)}\|(u,v)\|^{q-1}_{X(t)},\\
\|\tilde{u}(\tau,\cdot)\|^{q-1}_{L^{2q}(\mathbb{R}^n)} &\lesssim (1+B(\tau,0))^{\left(-\frac{n}{2m_1}+\frac{n}{4q}\right)(q-1)}\|(\tilde{u},\tilde{v})\|^{q-1}_{X(t)}.
\end{aligned}$$

Using the last estimates, in the same way we derived the estimates (2.35) to (2.38), we may conclude (2.30). The proof is completed. □

Now, we are interested in the case where one of the exponents p or q does not satisfy the conditions (2.27) or (2.28). For simplicity and without loss of generality we take $p<q$ and $p \leq \frac{m_2}{m_1}+\frac{2m_2}{n}$. Then we can prove a global (in time) existence result with loss of decay in one of the components of the solution and an interaction condition between the exponents p and q.

Theorem 2.6. *Let the data (u_0,u_1) and (v_0,v_1) belong to $\mathcal{A}_{m_1,s}\times\mathcal{A}_{m_2,s}$, $s\in(0,1)$, and $m_1,\, m_2\in[1,2]$. Moreover, we assume the following conditions for the exponents p and q:*

$$\frac{2}{m_1} \leq p < \frac{m_2}{m_1}+\frac{2m_2}{n}, \qquad \frac{2}{m_2} \leq q \leq p_{GN,s}(n), \tag{2.39}$$

and

$$\frac{n}{2} > m_2\Big(\frac{q+1}{pq-1}\Big). \tag{2.40}$$

Then, there exists a small constant ϵ_0 such that if

$$\|(u_0,u_1)\|_{\mathcal{A}_{m_1,s}}+\|(v_0,v_1)\|_{\mathcal{A}_{m_2,s}} \leq \epsilon_0,$$

then there exists a uniquely determined globally (in time) Sobolev solution to (2.1) in

$$\big(\mathcal{C}([0,\infty),H^s(\mathbb{R}^n))\big)^2.$$

Furthermore, the solution satisfies the following estimates:

$$
\begin{aligned}
\|u(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \big(1+B(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m_1}-\frac{1}{2}\right)+\gamma_{n,m_1,m_2}(p)} \\
&\qquad\qquad \times \left(\|(u_0,u_1)\|_{\mathcal{A}_{m_1,s}}+\|(v_0,v_1)\|_{\mathcal{A}_{m_2,s}}\right), \\
\big\||D|^s u(t,\cdot)\big\|_{L^2(\mathbb{R}^n)} &\lesssim \big(1+B(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m_1}-\frac{1}{2}\right)-\frac{s}{2}+\gamma_{n,m_1,m_2}(p)} \\
&\qquad\qquad \times \left(\|(u_0,u_1)\|_{\mathcal{A}_{m_1,s}}+\|(v_0,v_1)\|_{\mathcal{A}_{m_2,s}}\right), \\
\|v(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \big(1+B(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m_2}-\frac{1}{2}\right)} \left(\|(u_0,u_1)\|_{\mathcal{A}_{m_1,s}}+\|(v_0,v_1)\|_{\mathcal{A}_{m_2,s}}\right), \\
\big\||D|^s v(t,\cdot)\big\|_{L^2(\mathbb{R}^n)} &\lesssim \big(1+B(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m_2}-\frac{1}{2}\right)-\frac{s}{2}} \left(\|(u_0,u_1)\|_{\mathcal{A}_{m_1,s}}+\|(v_0,v_1)\|_{\mathcal{A}_{m_2,s}}\right),
\end{aligned}
$$

where $\gamma_{n,m_1,m_2}(p) = -\frac{n}{2m_2}p + \frac{n}{2m_1} + 1$ *is the loss of decay in comparison with the corresponding decay estimates for the solution* u *to the linear Cauchy problem with vanishing right-hand side.*

Remark 2.7.
- *If* $p = \frac{m_2}{m_1} + \frac{2m_2}{n}$*, then the loss of decay* $\gamma_{n,m_1,m_2}(p) = \varepsilon$ *with an arbitrarily small positive* ε*.*
- *The dimension* n *can be fixed in the particular case we are interested in. It depends on several parameters and conditions appearing in the statement of the theorem.*

Example 2.4. *If we consider the model (2.1) for* $n = 3$*,* $p = 1.55$ *and* $q = 2.6$*, then one possibility to satisfy all the conditions of the theorem is to choose* $m_1 = \frac{7}{4}$*,* $m_2 = \frac{5}{4}$ *and the regularity parameter* $s = 0.95$*.*

Sketch of the proof: To prove this theorem we follow the same steps as in the proof of Theorem 2.5 with a modification in the definition of the norm of the solution space $X(t)$. This modification generates a loss of decay.
Let

$$\|(u,v)\|_{X(t)} = \sup_{\tau\in[0,t]} \left\{\big(1+B(\tau,0)\big)^{-\gamma_{n,m_1,m_2}(p)} M_1(\tau,u) + M_2(\tau,v)\right\},$$

and N be the mapping which is defined in (2.11). Then our goal is to prove the inequalities (2.29) and (2.30). To estimate the nonlinear terms appearing in the first inequality we begin with u^{nl}. Using (2.33) and (2.34) in (2.32) we can get in the same way as we derived (2.15) the following estimate:

$$\big\||D|^s u^{nl}(t,\cdot)\big\|_{L^2(\mathbb{R}^n)} \lesssim (1+B(t,0))^{-\frac{n}{2}\left(\frac{1}{m_1}-\frac{1}{2}\right)-\frac{s}{2}+\gamma_{n,m_1,m_2}(p)} \|(u,v)\|_{X(t)}^p. \tag{2.41}$$

Analogously, we have

$$\big\|u^{nl}(t,\cdot)\big\|_{L^2(\mathbb{R}^n)} \lesssim (1+B(t,0))^{-\frac{n}{2}\left(\frac{1}{m_1}-\frac{1}{2}\right)+\gamma_{n,m_1,m_2}(p)} \|(u,v)\|_{X(t)}^p, \tag{2.42}$$

where we require (2.39), due to the use of the Gagliardo-Nirenberg inequality.
To estimate v^{nl} we obtain for $0 \le \tau \le t$ and after using the Gagliardo-Nirenberg inequality

$$\||u(\tau,\cdot)|^q\|_{L^{m_2}(\mathbb{R}^n)} \lesssim (1+B(\tau,0))^{-\frac{n}{2m_1}q+\frac{n}{2m_2}+\gamma_{n,m_1,m_2}(p)q} \|(u,v)\|_{X(t)}^q,$$

and
$$\||u(\tau,\cdot)|^q\|_{L^2(\mathbb{R}^n)} \lesssim (1+B(\tau,0))^{-\frac{n}{2m_1}q+\frac{n}{4}+\gamma_{n,m_1,m_2}(p)q}\|(u,v)\|^q_{X(t)}.$$
Using the last estimates leads to
$$\begin{aligned} \big\||D|^s v^{nl}(t,\cdot)\big\|_{L^2(\mathbb{R}^n)} &\lesssim \|(u,v)\|^q_{X(t)}(1+B(t,0))^{-\frac{n}{2}\left(\frac{1}{m_2}-\frac{1}{2}\right)-\frac{s}{2}}, && (2.43)\\ \big\|v^{nl}(t,\cdot)\big\|_{L^2(\mathbb{R}^n)} &\lesssim \|(u,v)\|^q_{X(t)}(1+B(t,0))^{-\frac{n}{2}\left(\frac{1}{m_2}-\frac{1}{2}\right)}, && (2.44)\end{aligned}$$
where we have taken account of $-\frac{n}{2m_1}+\frac{n}{2m_2}+\gamma_{n,m_1,m_2}(p)q+1<0$ which is equivalent to (2.40).
Finally, from (2.41) to (2.44) we complete the proof of (2.29). For the second inequality (2.30) we complete the proof in same way as we did in the proof of Theorem 2.5 after taking into consideration the new norm of the solution space $X(t)$ and all the assumptions, in particular, the interaction condition (2.40).

Remark 2.8. *If we are interested in the case*
$$\frac{2}{m_2}\le q<\frac{m_1}{m_2}+\frac{2m_1}{n} \quad \textit{and} \quad \frac{2}{m_1}\le p\le p_{GN,s}(n),$$
then the loss of decay will appear in the second component v under the following interaction condition
$$\frac{n}{2}>m_1\Big(\frac{p+1}{pq-1}\Big).$$

Remark 2.9. *From all the theorems of Section 2.1 we conclude that the admissible range of the regularity parameter s we can treat is defined by the following conditions:*
$$s\ge\max\left\{\frac{n(2-m_1)}{4};\frac{n(2-m_2)}{4}\right\},$$
and
$$s>\max\left\{\frac{n}{2}-\frac{m_1n^2}{nm_2+2m_1m_2};\frac{n}{2}-\frac{m_2n^2}{nm_1+2m_1m_2}\right\}.$$

Example 2.5. *Let the dimension $n=1$ and $m_1=m_2=1$, then from the last remark we can treat the model (2.1) if the regularity parameter satisfies $s\ge\frac{1}{4}$.*

2.2. Data from energy space

In this section we are interested in the "upper limit case" of Section 2.1, this means, in the case $s=1$. As we did before, we compare the exponents p and q with the modified Fujita exponent. We expect similar results to the case $s\in(0,1)$, but now the data has a larger regularity which allows for defining energy solutions and to introduce the terms $\|u_t(\tau,\cdot)\|_{L^2(\mathbb{R}^n)}$ and $\|v_t(\tau,\cdot)\|_{L^2(\mathbb{R}^n)}$ in the norm of the solution space $X(t)$. For this reason we will derive in this section also estimates for u_t and v_t, respectively, by controlling of all terms appearing in the norm of the solution space $X(t)$. First we turn to the case $p=q$.

2.2.1. The orders of power nonlinearities are above the modified Fujita exponent

Theorem 2.7. *Let $1 \leq n \leq 6$ and let us assume that the data (u_0, u_1) and (v_0, v_1) belong to $\mathcal{A}_{m,1} \times \mathcal{A}_{m,1}$ with $m \in [1, 2)$. Moreover, let*

$$p > p_{Fuj,m}(n), \qquad \begin{array}{ll} \frac{2}{m} \leq p < \infty, & \text{if} \quad n = 1, 2, \\ \frac{2}{m} \leq p \leq p_{GN}(n) & \text{if} \quad 2 < n \leq 6. \end{array} \tag{2.45}$$

Then, there exists a small constant ϵ_0 such that if

$$\|(u_0, u_1)\|_{\mathcal{A}_{m,1}} + \|(v_0, v_1)\|_{\mathcal{A}_{m,1}} \leq \epsilon_0,$$

then there exists a uniquely determined globally (in time) energy solution to (2.3) in

$$\left(\mathcal{C}([0, \infty), H^1(\mathbb{R}^n)) \cap \mathcal{C}^1([0, \infty), L^2(\mathbb{R}^n))\right)^2.$$

Furthermore, the solution satisfies the following decay estimates:

$$\begin{aligned}
\|u(t, \cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \big(1 + B(t, 0)\big)^{-\frac{n}{2}\left(\frac{1}{m} - \frac{1}{2}\right)} \left(\|(u_0, u_1)\|_{\mathcal{A}_{m,1}} + \|(v_0, v_1)\|_{\mathcal{A}_{m,1}}\right), \\
\big\|\nabla u(t, \cdot)\big\|_{L^2(\mathbb{R}^n)} &\lesssim \big(1 + B(t, 0)\big)^{-\frac{n}{2}\left(\frac{1}{m} - \frac{1}{2}\right) - \frac{1}{2}} \left(\|(u_0, u_1)\|_{\mathcal{A}_{m,1}} + \|(v_0, v_1)\|_{\mathcal{A}_{m,1}}\right), \\
\big\|u_t(t, \cdot)\big\|_{L^2(\mathbb{R}^n)} &\lesssim b(t)^{-1}\big(1 + B(t, 0)\big)^{-\frac{n}{2}\left(\frac{1}{m} - \frac{1}{2}\right) - 1} \left(\|(u_0, u_1)\|_{\mathcal{A}_{m,1}} + \|(v_0, v_1)\|_{\mathcal{A}_{m,1}}\right), \\
\|v(t, \cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \big(1 + B(t, 0)\big)^{-\frac{n}{2}\left(\frac{1}{m} - \frac{1}{2}\right)} \left(\|(u_0, u_1)\|_{\mathcal{A}_{m,1}} + \|(v_0, v_1)\|_{\mathcal{A}_{m,1}}\right), \\
\big\|\nabla v(t, \cdot)\big\|_{L^2(\mathbb{R}^n)} &\lesssim \big(1 + B(t, 0)\big)^{-\frac{n}{2}\left(\frac{1}{m} - \frac{1}{2}\right) - \frac{1}{2}} \left(\|(u_0, u_1)\|_{\mathcal{A}_{m,1}} + \|(v_0, v_1)\|_{\mathcal{A}_{m,1}}\right), \\
\big\|v_t(t, \cdot)\big\|_{L^2(\mathbb{R}^n)} &\lesssim b(t)^{-1}\big(1 + B(t, 0)\big)^{-\frac{n}{2}\left(\frac{1}{m} - \frac{1}{2}\right) - 1} \left(\|(u_0, u_1)\|_{\mathcal{A}_{m,1}} + \|(v_0, v_1)\|_{\mathcal{A}_{m,1}}\right).
\end{aligned}$$

Proof. The proof can be completed by following the same steps of the proof of Theorem 1.9, by defining a modified solution space $X(t)$ with a norm consisting of two parts with respect to u and v. We control the components of the norms $M(\tau, u)$ and $M(\tau, v)$ of the solution space separately because there is no any interaction between the power nonlinearities. □

In the following corollary we present a result for the case $m = 1$ which is a generalization to systems of the result which was proved in [4] for one single equation.

Corollary 2.2. *Let $1 \leq n \leq 4$ and*

$$\begin{array}{cll} p > p_{Fuj}(n) & \text{if} & n = 1, 2, \\ 2 \leq p \leq 3 = p_{GN}(3) & \text{if} & n = 3, \\ p = 2 = p_{GN}(4) & \text{if} & n = 4. \end{array} \tag{2.46}$$

Let the data (u_0, u_1) and (v_0, v_1) belong to $\mathcal{A}_{1,1} \times \mathcal{A}_{1,1}$. Then, there exists a small constant ϵ_0 such that if

$$\|(u_0, u_1)\|_{\mathcal{A}_{1,1}} + \|(v_0, v_1)\|_{\mathcal{A}_{1,1}} \leq \epsilon_0,$$

then there exists a uniquely determined globally (in time) energy solution to (2.3) in

$$\big(\mathcal{C}([0,\infty),H^1(\mathbb{R}^n))\cap\mathcal{C}^1([0,\infty),L^2(\mathbb{R}^n))\big)^2.$$

Furthermore, the solution satisfies the following decay estimates:

$$\begin{aligned}
\|u(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \big(1+B(t,0)\big)^{-\frac{n}{4}}\left(\|(u_0,u_1)\|_{\mathcal{A}_{1,1}}+\|(v_0,v_1)\|_{\mathcal{A}_{1,1}}\right),\\
\|v(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \big(1+B(t,0)\big)^{-\frac{n}{4}}\left(\|(u_0,u_1)\|_{\mathcal{A}_{1,1}}+\|(v_0,v_1)\|_{\mathcal{A}_{1,1}}\right),\\
\big\|\nabla u(t,\cdot)\big\|_{L^2(\mathbb{R}^n)} &\lesssim \big(1+B(t,0)\big)^{-\frac{n}{4}-\frac{1}{2}}\left(\|(u_0,u_1)\|_{\mathcal{A}_{1,1}}+\|(v_0,v_1)\|_{\mathcal{A}_{1,1}}\right),\\
\big\|\nabla v(t,\cdot)\big\|_{L^2(\mathbb{R}^n)} &\lesssim \big(1+B(t,0)\big)^{-\frac{n}{4}-\frac{1}{2}}\left(\|(u_0,u_1)\|_{\mathcal{A}_{1,1}}+\|(v_0,v_1)\|_{\mathcal{A}_{1,1}}\right),\\
\big\|u_t(t,\cdot)\big\|_{L^2(\mathbb{R}^n)} &\lesssim b(t)^{-1}\big(1+B(t,0)\big)^{-\frac{n}{4}-1}\left(\|(u_0,u_1)\|_{\mathcal{A}_{1,1}}+\|(v_0,v_1)\|_{\mathcal{A}_{1,1}}\right),\\
\big\|v_t(t,\cdot)\big\|_{L^2(\mathbb{R}^n)} &\lesssim b(t)^{-1}\big(1+B(t,0)\big)^{-\frac{n}{4}-1}\left(\|(u_0,u_1)\|_{\mathcal{A}_{1,1}}+\|(v_0,v_1)\|_{\mathcal{A}_{1,1}}\right).
\end{aligned}$$

If $p\neq q$, then we get the following result.

Theorem 2.8. *Let* $1\leq n\leq 6$, $(u_0,u_1),(v_0,v_1)\in\mathcal{A}_{m,1}\times\mathcal{A}_{m,1}$, $m\in[1,2)$ *and let*

$$\min\{p;q\}>p_{Fuj,m}(n),\qquad \begin{array}{rl} \frac{2}{m}\leq\min\{p;q\}<\infty & \text{if}\quad n=1,2,\\ \frac{2}{m}\leq\min\{p;q\}<\max\{p;q\}\leq p_{GN}(n) & \text{if}\quad 2<n\leq 6.\end{array}$$

Then, there exists a small constant ϵ_0 *such that if*

$$\|(u_0,u_1)\|_{\mathcal{A}_{m,1}}+\|(v_0,v_1)\|_{\mathcal{A}_{m,1}}\leq\epsilon_0,$$

then there exists a uniquely determined globally (in time) energy solution to (2.1) in

$$\big(\mathcal{C}([0,\infty),H^1(\mathbb{R}^n))\cap\mathcal{C}^1([0,\infty),L^2(\mathbb{R}^n))\big)^2.$$

Furthermore, the solution satisfies the same decay estimates of Theorem 2.7.

2.2.2. Only one exponent is above the modified Fujita exponent

In this case we assume that one exponent is below the modified Fujita exponent. This generates a loss of decay. This loss of decay implies an interaction between the power nonlinearities which is described by condition (2.48).

Theorem 2.9. *Let* $n\leq\frac{2m^2}{2-m}$ *and* $n<\frac{2m}{m-1}$. *The data* (u_0,u_1) *and* (v_0,v_1) *are supposed to belong to* $\mathcal{A}_{m,1}\times\mathcal{A}_{m,1}$ *with* $m\in[1,2)$. *Moreover, let*

$$\begin{array}{rl} \frac{2}{m}\leq p<p_{Fuj,m}(n)<q<\infty & \text{if}\quad n\leq 2,\\ \frac{2}{m}\leq p<p_{Fuj,m}(n)<q\leq p_{GN}(n) & \text{if}\quad n>2,\end{array}\tag{2.47}$$

and

$$\frac{n}{2}>m\Big(\frac{q+1}{pq-1}\Big).\tag{2.48}$$

Then, there exists a small constant ϵ_0 such that if

$$\|(u_0,u_1)\|_{\mathcal{A}_{m,1}}+\|(v_0,v_1)\|_{\mathcal{A}_{m,1}}\leq \epsilon_0,$$

then there exists a uniquely determined globally (in time) energy solution to (2.1) in

$$\big(\mathcal{C}([0,\infty),H^1(\mathbb{R}^n))\cap\mathcal{C}^1([0,\infty),L^2(\mathbb{R}^n))\big)^2.$$

Furthermore, the solution satisfies the following estimates:

$$\begin{aligned}
\|u(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \big(1+B(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)+\gamma_{n,m,\varepsilon}(p)}\left(\|(u_0,u_1)\|_{\mathcal{A}_{m,1}}+\|(v_0,v_1)\|_{\mathcal{A}_{m,1}}\right),\\
\|\nabla u(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \big(1+B(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{1}{2}+\gamma_{n,m,\varepsilon}(p)}\\
&\qquad\qquad\times\left(\|(u_0,u_1)\|_{\mathcal{A}_{m,1}}+\|(v_0,v_1)\|_{\mathcal{A}_{m,1}}\right),\\
\|u_t(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim b(t)^{-1}\big(1+B(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-1+\gamma_{n,m,\varepsilon}(p)}\\
&\qquad\qquad\times\left(\|(u_0,u_1)\|_{\mathcal{A}_{m,1}}+\|(v_0,v_1)\|_{\mathcal{A}_{m,1}}\right),\\
\|v(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \big(1+B(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)}\left(\|(u_0,u_1)\|_{\mathcal{A}_{m,1}}+\|(v_0,v_1)\|_{\mathcal{A}_{m,1}}\right),\\
\|\nabla v(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \big(1+B(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{1}{2}}\left(\|(u_0,u_1)\|_{\mathcal{A}_{m,1}}+\|(v_0,v_1)\|_{\mathcal{A}_{m,1}}\right),\\
\|v_t(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim b(t)^{-1}\big(1+B(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-1}\left(\|(u_0,u_1)\|_{\mathcal{A}_{m,1}}+\|(v_0,v_1)\|_{\mathcal{A}_{m,1}}\right),
\end{aligned}$$

where $\gamma_{n,m,\varepsilon}(p)=-\frac{n}{2m}(p-1)+1+\varepsilon$ represents the loss of decay in comparison with the corresponding decay estimates for the solution u to the linear Cauchy problem with vanishing right-hand side.

Remark 2.10. *From the last theorem we conclude that if we choose $m=1$, then we get only $n=1,2$ and if we choose $m=2$, then we get $n=1,2,3$. In other words, the freedom of the choice of m between 1 and 2 allows us to treat a larger dimension n. The following example shows a particular case.*

Example 2.6. *If we have $p=\frac{8}{5}$ and $q=\frac{5}{3}$, then in order to get a global (in time) existence result in dimension $n=5$ we choose $m=\frac{3}{2}$. With this choice the conditions (2.47) and (2.48) are satisfied.*

Proof. We follow the same steps of the proof of Theorem 2.3 by taking $s=1$, and add the terms

$$b(\tau)(1+B(\tau,0))^{\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)+1}\|u_t(\tau,\cdot)\|_{L^2(\mathbb{R}^n)},$$

and

$$b(\tau)(1+B(\tau,0))^{\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)+1}\|v_t(\tau,\cdot)\|_{L^2(\mathbb{R}^n)}$$

in the definition of the norm of the solution space

$$X(t)=\big\{(u,v)\in\big(\mathcal{C}([0,t],H^1(\mathbb{R}^n))\cap\mathcal{C}^1([0,t],L^2(\mathbb{R}^n))\big)^2\big\}.$$

It remains to prove in the existence part the estimates

$$\big\|u_t^{nl}(t,\cdot)\big\|_{L^2(\mathbb{R}^n)}\lesssim b(t)^{-1}(1+B(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-1+\gamma_{n,m,\varepsilon}(p)}\|(u,v)\|^p_{X(t)}, \tag{2.49}$$

and

$$\big\|v_t^{nl}(t,\cdot)\big\|_{L^2(\mathbb{R}^n)} \lesssim b(t)^{-1}(1+B(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-1}\|(u,v)\|^q_{X(t)}. \tag{2.50}$$

To prove the estimate (2.49) we obtain from the estimate (1.23) of Theorem 1.4, with $m=2$ for $\tau\in[\frac{t}{2},t]$, the inequality

$$\begin{aligned}
&\big\|u_t^{nl}(t,\cdot)\big\|_{L^2(\mathbb{R}^n)}\\
&\quad\lesssim \int_0^{\frac{t}{2}} b(t)^{-1}b(\tau)^{-1}(1+B(t,\tau))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-1}\||v(\tau,x)|^p\|_{L^m(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)}d\tau\\
&\qquad+\int_{\frac{t}{2}}^t b(t)^{-1}b(\tau)^{-1}(1+B(t,\tau))^{-1}\||v(\tau,x)|^p\|_{L^2(\mathbb{R}^n)}d\tau.
\end{aligned}$$

If $\tau\in[0,\frac{t}{2}]$, then after using the Gagliardo-Nirenberg inequality we arrive at

$$\begin{aligned}
&\int_0^{\frac{t}{2}} b(t)^{-1}b(\tau)^{-1}(1+B(t,\tau))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-1}\||v(\tau,x)|^p\|_{L^m(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)}d\tau\\
&\quad\lesssim \|(u,v)\|^p_{X(t)}\int_0^{\frac{t}{2}} b(t)^{-1}b(\tau)^{-1}(1+B(t,\tau))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-1}(1+B(\tau,0))^{-\frac{n}{2m}p+\frac{n}{2m}}d\tau\\
&\quad\lesssim \|(u,v)\|^p_{X(t)}b(t)^{-1}(1+B(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-1}\int_0^{\frac{t}{2}} b(\tau)^{-1}(1+B(\tau,0))^{-\frac{n}{2m}p+\frac{n}{2m}}d\tau\\
&\quad\lesssim b(t)^{-1}(1+B(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-1-\frac{n}{2m}(p-1)+1}\|(u,v)\|^p_{X(t)}\\
&\quad\lesssim b(t)^{-1}(1+B(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-1+\gamma_{n,m,\varepsilon}(p)}\|(u,v)\|^p_{X(t)}.
\end{aligned}$$

If $\tau\in[\frac{t}{2},t]$, then again the use of the Gagliardo-Nirenberg inequality leads to

$$\begin{aligned}
&\int_{\frac{t}{2}}^t b(t)^{-1}b(\tau)^{-1}(1+B(t,\tau))^{-1}\||v(\tau,x)|^p\|_{L^2(\mathbb{R}^n)}d\tau\\
&\quad\lesssim \|(u,v)\|^p_{X(t)}b(t)^{-1}(1+B(t,0))^{-\frac{n}{2m}p+\frac{n}{4}}\int_{\frac{t}{2}}^t b(\tau)^{-1}(1+B(t,\tau))^{-1}d\tau\\
&\quad\lesssim \|(u,v)\|^p_{X(t)}b(t)^{-1}(1+B(t,0))^{-\frac{n}{2m}p+\frac{n}{4}}\log(1+B(t,0))\\
&\quad\lesssim \|(u,v)\|^p_{X(t)}b(t)^{-1}(1+B(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-1+\gamma_{n,m,\varepsilon}(p)-\varepsilon}\log(1+B(t,0))\\
&\quad\lesssim b(t)^{-1}(1+B(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-1+\gamma_{n,m,\varepsilon}(p)}\|(u,v)\|^p_{X(t)}.
\end{aligned}$$

Then, (2.49) is proved.

Now we prove (2.50). From the estimate (1.23) of Theorem 1.4, with $m=2$ for $\tau\in[\frac{t}{2},t]$, we get

$$\begin{aligned}
&\big\|v_t^{nl}(t,\cdot)\big\|_{L^2(\mathbb{R}^n)}\\
&\quad\lesssim \int_0^{\frac{t}{2}} b(t)^{-1}b(\tau)^{-1}(1+B(t,\tau))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-1}\||u(\tau,\cdot)|^q\|_{L^m(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)}d\tau\\
&\qquad+\int_{\frac{t}{2}}^t b(t)^{-1}b(\tau)^{-1}(1+B(t,\tau))^{-1}\||u(\tau,\cdot)|^q\|_{L^2(\mathbb{R}^n)}d\tau.
\end{aligned}$$

If $\tau \in [0, \frac{t}{2}]$, then after using the Gagliardo-Nirenberg inequality it follows

$$\begin{aligned}
&\int_0^{\frac{t}{2}} b(t)^{-1}b(\tau)^{-1}(1+B(t,\tau))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-1}\||u(\tau,\cdot)|^q\|_{L^m(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)}d\tau\\
&\lesssim \|(u,v)\|_{X(t)}^q\int_0^{\frac{t}{2}} b(t)^{-1}b(\tau)^{-1}(1+B(t,\tau))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-1}(1+B(\tau,0))^{-\frac{n}{2m}q+\frac{n}{2m}+\gamma_{n,m,\varepsilon}(p)q}d\tau\\
&\lesssim \|(u,v)\|_{X(t)}^q b(t)^{-1}(1+B(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-1}\int_0^{\frac{t}{2}} b(\tau)^{-1}(1+B(\tau,0))^{-\frac{n}{2m}q+\frac{n}{2m}+\gamma_{n,m,\varepsilon}(p)q}d\tau\\
&\lesssim b(t)^{-1}(1+B(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-1}\|(u,v)\|_{X(t)}^q,
\end{aligned}$$

where we take account of

$$-\frac{n}{2m}q+\frac{n}{2m}+\gamma_{n,m,\varepsilon}(p)q<-1$$

which is equivalent to (2.48).
If $\tau \in [\frac{t}{2}, t]$, then using the Gagliardo-Nirenberg inequality again gives

$$\begin{aligned}
&\int_{\frac{t}{2}}^t b(t)^{-1}b(\tau)^{-1}(1+B(t,\tau))^{-1}\||u(\tau,\cdot)|^q\|_{L^2(\mathbb{R}^n)}d\tau\\
&\qquad\lesssim \|(u,v)\|_{X(t)}^q b(t)^{-1}(1+B(t,0))^{-\frac{n}{2m}q+\frac{n}{4}+\gamma_{n,m,\varepsilon}(p)q}\int_{\frac{t}{2}}^t b(\tau)^{-1}(1+B(t,\tau))^{-1}d\tau\\
&\qquad\lesssim \|(u,v)\|_{X(t)}^q b(t)^{-1}(1+B(t,0))^{-\frac{n}{2m}q+\frac{n}{4}+\gamma_{n,m,\varepsilon}(p)q}\log(1+B(t,0))\\
&\qquad\lesssim \|(u,v)\|_{X(t)}^q b(t)^{-1}(1+B(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{n}{2m}q+\frac{n}{2m}+\gamma_{n,m,\varepsilon}(p)q+\varepsilon}\log(1+B(t,0))\\
&\qquad\lesssim b(t)^{-1}(1+B(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-1}\|(u,v)\|_{X(t)}^q,
\end{aligned}$$

where we use

$$-\frac{n}{2m}q+\frac{n}{2m}+\gamma_{n,m,\varepsilon}(p)q+\varepsilon<0$$

which is equivalent to (2.48) for sufficiently small ε. Then, (2.50) is proved.
For the uniqueness part we use the estimates (2.20) to (2.23) with $s=1$ and we prove in a similar way to (2.49) and (2.50) the following estimates:

$$\begin{aligned}
&\left\|\partial_t\int_0^t E_1(t,\tau,x)*_{(x)}\left(|v(\tau,x)|^p-|\tilde{v}(\tau,x)|^p\right)d\tau\right\|_{L^2(\mathbb{R}^n)}\\
&\quad\lesssim b(t)^{-1}(1+B(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-1+\gamma_{n,m,\varepsilon}(p)}\\
&\qquad\qquad\times\sup_{\tau\in[0,t]}M(\tau,v-\tilde{v})\left(\|(u,v)\|_{X(t)}^{p-1}+\|(\tilde{u},\tilde{v})\|_{X(t)}^{p-1}\right),
\end{aligned}$$

and

$$\begin{aligned}
&\left\|\partial_t\int_0^t E_1(t,\tau,x)*_{(x)}\left(|u(\tau,x)|^q-|\tilde{u}(\tau,x)|^q\right)d\tau\right\|_{L^2(\mathbb{R}^n)}\\
&\quad\lesssim b(t)^{-1}(1+B(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-1}\\
&\qquad\qquad\times\sup_{\tau\in[0,t]}M(\tau,u-\tilde{u})\left(\|(u,v)\|_{X(t)}^{q-1}+\|(\tilde{u},\tilde{v})\|_{X(t)}^{q-1}\right).
\end{aligned}$$

The proof is completed. $\square$

The statement of Remark 2.3 remains still true in the case $p = p_{Fuj,m}(n)$, this means, that the loss of decay $\gamma_{n,m,\varepsilon}(p) = \varepsilon$.

Remark 2.11. *In Theorem 2.3 the data belong to $\mathcal{A}_{m,s}$ with $0 < s < 1$. The exponents p and q generate a loss of decay $\gamma_{n,m}(p) = -\frac{n}{2m}(p-1)+1$. We remark that this loss of decay perturbed by a positive small number ε appears in the statements of Theorem 2.9 in the limit case $s = 1$ when the data belong to $\mathcal{A}_{m,1}$.*

If we change the roles of p and q, then we get the following result.

Theorem 2.10. *Let $n \leq \frac{2m^2}{2-m}$ and $n < \frac{2m}{m-1}$. The data (u_0, u_1) and (v_0, v_1) are supposed to belong to $\mathcal{A}_{m,1} \times \mathcal{A}_{m,1}$ with $m \in [1, 2)$. Moreover, let the exponents p and q satisfy the following conditions:*

$$\begin{array}{ll} \frac{2}{m} \leq q < p_{Fuj,m}(n) < p < \infty & \text{if} \quad n \leq 2, \\ \frac{2}{m} \leq q < p_{Fuj,m}(n) < p \leq p_{GN}(n) & \text{if} \quad n > 2, \end{array} \tag{2.51}$$

and

$$\frac{n}{2} > m\Big(\frac{p+1}{pq-1}\Big). \tag{2.52}$$

Then, there exists a small constant ϵ_0 such that if

$$\|(u_0, u_1)\|_{\mathcal{A}_{m,1}} + \|(v_0, v_1)\|_{\mathcal{A}_{m,1}} \leq \epsilon_0,$$

then there exists a uniquely determined globally (in time) energy solution to (2.1) in

$$\big(\mathcal{C}([0,\infty), H^1(\mathbb{R}^n)) \cap \mathcal{C}^1([0,\infty), L^2(\mathbb{R}^n)\big)^2.$$

Furthermore, the solution satisfies the following estimates:

$$\begin{aligned}
\|u(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \big(1+B(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)} \big(\|(u_0,u_1)\|_{\mathcal{A}_{m,1}} + \|(v_0,v_1)\|_{\mathcal{A}_{m,1}}\big),\\
\|\nabla u(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \big(1+B(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{1}{2}} \big(\|(u_0,u_1)\|_{\mathcal{A}_{m,1}} + \|(v_0,v_1)\|_{\mathcal{A}_{m,1}}\big),\\
\|u_t(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim b(t)^{-1}\big(1+B(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-1} \big(\|(u_0,u_1)\|_{\mathcal{A}_{m,1}} + \|(v_0,v_1)\|_{\mathcal{A}_{m,1}}\big),\\
\|v(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \big(1+B(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)+\gamma_{n,m,\varepsilon}(q)} \big(\|(u_0,u_1)\|_{\mathcal{A}_{m,1}} + \|(v_0,v_1)\|_{\mathcal{A}_{m,1}}\big),\\
\|\nabla v(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \big(1+B(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{1}{2}+\gamma_{n,m,\varepsilon}(q)} \big(\|(u_0,u_1)\|_{\mathcal{A}_{m,1}} + \|(v_0,v_1)\|_{\mathcal{A}_{m,1}}\big),\\
\|v_t(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim b(t)^{-1}\big(1+B(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-1+\gamma_{n,m,\varepsilon}(q)}\\
&\qquad \times \big(\|(u_0,u_1)\|_{\mathcal{A}_{m,1}} + \|(v_0,v_1)\|_{\mathcal{A}_{m,1}}\big),
\end{aligned}$$

where $\gamma_{n,m,\varepsilon}(q) = -\frac{n}{2m}(q-1)+1+\varepsilon$ represents the loss of decay in comparison with the corresponding decay estimates for the solution v to the linear Cauchy problem with vanishing right-hand side.

The statements of the Theorems 2.9 and 2.10 yield the following corollary.

Corollary 2.3. *Let $n \leq \frac{2m^2}{2-m}$ and $n < \frac{2m}{m-1}$. The data (u_0, u_1) and (v_0, v_1) are supposed to belong to $\mathcal{A}_{m,1} \times \mathcal{A}_{m,1}$ with $m \in [1, 2)$. Moreover, let the exponents p and q satisfy the following conditions:*

$$\begin{array}{ll} \frac{2}{m} \leq \min\{p;q\} \leq p_{Fuj,m}(n) < \max\{p;q\} < \infty & \textit{if} \quad n \leq 2, \\ \frac{2}{m} \leq \min\{p;q\} \leq p_{Fuj,m}(n) < \max\{p;q\} \leq p_{GN}(n) & \textit{if} \quad n > 2, \end{array}$$

and

$$\frac{n}{2} > m\Big(\frac{\max\{p;q\}+1}{pq-1}\Big).$$

Then, there exists a small constant ϵ_0 such that if

$$\|(u_0,u_1)\|_{\mathcal{A}_{m,1}} + \|(v_0,v_1)\|_{\mathcal{A}_{m,1}} \leq \epsilon_0,$$

then there exists a uniquely determined globally (in time) energy solution to (2.1) in

$$\big(\mathcal{C}([0,\infty), H^1(\mathbb{R}^n)) \cap \mathcal{C}^1([0,\infty), L^2(\mathbb{R}^n))\big)^2.$$

Furthermore, the solution satisfies the following estimates:

$$\begin{aligned}
\|u(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \big(1+B(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)+[\gamma_{n,m,\varepsilon}(p)]^+} \left(\|(u_0,u_1)\|_{\mathcal{A}_{m,1}} + \|(v_0,v_1)\|_{\mathcal{A}_{m,1}}\right), \\
\|\nabla u(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \big(1+B(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{1}{2}+[\gamma_{n,m,\varepsilon}(p)]^+} \\
&\qquad \times \left(\|(u_0,u_1)\|_{\mathcal{A}_{m,1}} + \|(v_0,v_1)\|_{\mathcal{A}_{m,1}}\right), \\
\|u_t(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim b(t)^{-1}\big(1+B(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-1+[\gamma_{n,m,\varepsilon}(p)]^+} \\
&\qquad \times \left(\|(u_0,u_1)\|_{\mathcal{A}_{m,1}} + \|(v_0,v_1)\|_{\mathcal{A}_{m,1}}\right), \\
\|v(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \big(1+B(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)+[\gamma_{n,m,\varepsilon}(q)]^+} \left(\|(u_0,u_1)\|_{\mathcal{A}_{m,1}} + \|(v_0,v_1)\|_{\mathcal{A}_{m,1}}\right), \\
\|\nabla v(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \big(1+B(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{1}{2}+[\gamma_{n,m,\varepsilon}(q)]^+} \\
&\qquad \times \left(\|(u_0,u_1)\|_{\mathcal{A}_{m,1}} + \|(v_0,v_1)\|_{\mathcal{A}_{m,1}}\right), \\
\|v_t(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim b(t)^{-1}\big(1+B(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-1+[\gamma_{n,m,\varepsilon}(q)]^+} \\
&\qquad \times \left(\|(u_0,u_1)\|_{\mathcal{A}_{m,1}} + \|(v_0,v_1)\|_{\mathcal{A}_{m,1}}\right),
\end{aligned}$$

where $[\gamma_{n,m,\varepsilon}(p)]^+ = \max\{\gamma_{n,m,\varepsilon}(p);0\}$ and $[\gamma_{n,m,\varepsilon}(q)]^+ = \max\{\gamma_{n,m,\varepsilon}(q);0\}$.

2.2.3. Different additional regularities

As we did at the end of Section 2.1 in this section we present some results for Cauchy data having different additional regularities $L^{m_1}(\mathbb{R}^n)$ and $L^{m_2}(\mathbb{R}^n)$. The proofs can be done by combining the results of Theorem 2.5 with Theorem 2.7 to prove Theorem 2.11 and Theorem 2.6 with Theorem 2.9 to prove Theorem 2.12.

Theorem 2.11. *Let $n \leq \max\{\frac{4}{2-m_1}; \frac{4}{2-m_2}\}$. The data (u_0, u_1) and (v_0, v_1) are supposed to belong to $\mathcal{A}_{m_1,1} \times \mathcal{A}_{m_2,1}$, $s \in (0,1)$ and $m_1, m_2 \in [1,2]$. Moreover, the exponents* *satisfy the following conditions:*

$$p > \frac{2m_2}{n} + \frac{m_2}{m_1}, \qquad \frac{2}{m_1} \leq p \leq p_{GN}(n), \tag{2.53}$$

$$q > \frac{2m_1}{n} + \frac{m_1}{m_2}, \qquad \frac{2}{m_2} \leq p \leq p_{GN}(n). \tag{2.54}$$

Then, there exists a small constant ϵ_0 *such that if*

$$\|(u_0,u_1)\|_{\mathcal{A}_{m_1,1}} + \|(v_0,v_1)\|_{\mathcal{A}_{m_2,1}} \leq \epsilon_0,$$

then there exists a uniquely determined globally (in time) energy solution to (2.1) in

$$\big(\mathcal{C}([0,\infty), H^1(\mathbb{R}^n)) \cap \mathcal{C}^1([0,\infty), L^2(\mathbb{R}^n))\big)^2.$$

Furthermore, the solution satisfies the following decay estimates:

$$\begin{aligned}
\|u(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \big(1+B(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m_1}-\frac{1}{2}\right)} \left(\|(u_0,u_1)\|_{\mathcal{A}_{m_1,1}} + \|(v_0,v_1)\|_{\mathcal{A}_{m_2,1}}\right),\\
\big\|\nabla u(t,\cdot)\big\|_{L^2(\mathbb{R}^n)} &\lesssim \big(1+B(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m_1}-\frac{1}{2}\right)-\frac{1}{2}}\\
&\qquad\qquad \times \left(\|(u_0,u_1)\|_{\mathcal{A}_{m_1,1}} + \|(v_0,v_1)\|_{\mathcal{A}_{m_2,1}}\right),\\
\|u_t(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim b(t)^{-1}\big(1+B(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m_1}-\frac{1}{2}\right)-1}\\
&\qquad\qquad \times \left(\|(u_0,u_1)\|_{\mathcal{A}_{m_1,1}} + \|(v_0,v_1)\|_{\mathcal{A}_{m_2,1}}\right),\\
\|v(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \big(1+B(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m_2}-\frac{1}{2}\right)} \left(\|(u_0,u_1)\|_{\mathcal{A}_{m_1,1}} + \|(v_0,v_1)\|_{\mathcal{A}_{m_2,1}}\right),\\
\big\|\nabla v(t,\cdot)\big\|_{L^2(\mathbb{R}^n)} &\lesssim \big(1+B(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m_2}-\frac{1}{2}\right)-\frac{1}{2}}\\
&\qquad\qquad \times \left(\|(u_0,u_1)\|_{\mathcal{A}_{m_1,1}} + \|(v_0,v_1)\|_{\mathcal{A}_{m_2,1}}\right),\\
\|v_t(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim b(t)^{-1}\big(1+B(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m_2}-\frac{1}{2}\right)-1}\\
&\qquad\qquad \times \left(\|(u_0,u_1)\|_{\mathcal{A}_{m_1,1}} + \|(v_0,v_1)\|_{\mathcal{A}_{m_2,1}}\right).
\end{aligned}$$

Proof. To prove this theorem we follow the same steps of the proof to Theorem 2.5, with additional terms appearing in the norm of the solution space. The conditions (2.53) and (2.54) come in after using the Gagliardo-Nirenberg inequality and controlling the nonlinear terms. □

Example 2.7. *Let us assume* $n = 2$*,* $m_1 = 2$ *and* $m_2 = 1$*. Then the admissible ranges for the exponents* p *and* q *of the power nonlinearities are as follows:*

$$p > \frac{3}{2}, \quad q > 4.$$

We can not make any choice for the parameters m_1 *and* m_2 *for* $n = 3$ *because the admissible range for* q *will be empty.*

Theorem 2.12. *Let* $n \leq \max\{\frac{4}{2-m_2}; \frac{2m_1m_2}{2-m_1}\}$*. The data* (u_0,u_1) *and* (v_0,v_1) *are supposed to belong to* $\mathcal{A}_{m_1,1} \times \mathcal{A}_{m_2,1}$*,* $s \in (0,1)$ *and* $m_1, m_2 \in [1,2]$*. Moreover, the exponents* p *and* q *satisfy the following conditions:*

$$\frac{2}{m_1} \leq p < \frac{m_2}{m_1} + \frac{2m_2}{n}, \qquad \frac{2}{m_2} \leq q \leq p_{GN,1}(n),$$

and

$$\frac{n}{2} < m_2\Big(\frac{q+1}{pq-1}\Big). \tag{2.55}$$

Then, there exists a small constant ϵ_0 such that if

$$\|(u_0,u_1)\|_{\mathcal{A}_{m_1,1}} + \|(v_0,v_1)\|_{\mathcal{A}_{m_2,1}} \le \epsilon_0,$$

then there exists a uniquely determined globally (in time) energy solution to (2.1) in

$$\big(\mathcal{C}([0,\infty),H^1(\mathbb{R}^n)) \cap \mathcal{C}^1([0,\infty),L^2(\mathbb{R}^n))\big)^2 .$$

Furthermore, the solution satisfies the following estimates:

$$\begin{aligned}
\|u(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \big(1+B(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m_1}-\frac{1}{2}\right)+\gamma_{n,m_1,m_2}(p)} \\
&\qquad \times \left(\|(u_0,u_1)\|_{\mathcal{A}_{m_1,1}} + \|(v_0,v_1)\|_{\mathcal{A}_{m_2,1}}\right),\\
\big\|\nabla u(t,\cdot)\big\|_{L^2(\mathbb{R}^n)} &\lesssim \big(1+B(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m_1}-\frac{1}{2}\right)-\frac{1}{2}+\gamma_{n,m_1,m_2}(p)} \\
&\qquad \times \left(\|(u_0,u_1)\|_{\mathcal{A}_{m_1,1}} + \|(v_0,v_1)\|_{\mathcal{A}_{m_2,1}}\right),\\
\|u_t(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim b(t)^{-1}\big(1+B(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m_1}-\frac{1}{2}\right)-1+\gamma_{n,m_1,m_2}(p)} \\
&\qquad \times \left(\|(u_0,u_1)\|_{\mathcal{A}_{m_1,1}} + \|(v_0,v_1)\|_{\mathcal{A}_{m_2,1}}\right),\\
\|v(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \big(1+B(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m_2}-\frac{1}{2}\right)} \left(\|(u_0,u_1)\|_{\mathcal{A}_{m_1,1}} + \|(v_0,v_1)\|_{\mathcal{A}_{m_2,1}}\right),\\
\big\|\nabla v(t,\cdot)\big\|_{L^2(\mathbb{R}^n)} &\lesssim \big(1+B(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m_2}-\frac{1}{2}\right)-\frac{1}{2}} \left(\|(u_0,u_1)\|_{\mathcal{A}_{m_1,1}} + \|(v_0,v_1)\|_{\mathcal{A}_{m_2,1}}\right),\\
\|v_t(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim b(t)^{-1}\big(1+B(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m_2}-\frac{1}{2}\right)-1} \\
&\qquad \times \left(\|(u_0,u_1)\|_{\mathcal{A}_{m_1,1}} + \|(v_0,v_1)\|_{\mathcal{A}_{m_2,1}}\right),
\end{aligned}$$

where

$$\gamma_{n,m_1,m_2}(p) = -\frac{n}{2m_2}p + \frac{n}{2m_1} + 1 + \varepsilon$$

represents the loss of decay of the component u of the solution in comparison with the corresponding decay estimates for the solution to the linear Cauchy problem for u with vanishing right-hand side.

Proof. To prove this theorem we follow the same steps of the proof to Theorem 2.6, with additional terms appearing in the norm of the solution space associated with the expected loss of decay. The condition (2.55) comes in as a result of the modified norm of the solution space $X(t)$ and the control of v^{nl}. □

2.3. Data from Sobolev spaces with suitable regularity

This section is devoted to the case, where the data are from Sobolev spaces with suitable regularity. From Chapter 1, Section 1.4 we remark that the admissible range for p and q was among other things defined by using the regularity parameter of the data. We divide this section into two parts:

• In the fist subsection we deal with the case where $\max\{p_{Fuj,m}(n);\lceil s\rceil;2\}$ is influenced by all of its components $p_{Fuj,m}(n)$, $\lceil s\rceil$ and 2. Results are given in Theorem 2.13.

• In the second subsection we treat the Cauchy problem under the assumption

$$\max\big\{p_{Fuj,m}(n);\lceil s\rceil;2\big\}=\lceil s\rceil,$$

where $n\geq 4$ and $s\geq 3$.

2.3.1. The orders of power nonlinearities and the regularity of data coincide

Firstly, we study the case $s_1=s_2=s<p=q$. The following theorem can be concluded immediately from Theorem 1.11.

Theorem 2.13. *Let $n\geq 3$ and $s\in(1,3)$ be a non-integer number. The data are supposed to satisfy $(u_0,u_1),\,(v_0,v_1)\in\mathcal{A}_{m,s,\varepsilon}\times\mathcal{A}_{m,s,\varepsilon}$, for $m\in[1,2)$. Finally, the following conditions are satisfied for the exponent p:*

$$\begin{array}{llll}\max\big\{p_{Fuj,m}(n);\lceil s\rceil;2\big\}< & p & <\infty & \text{if}\quad s\in[\frac{n}{2},\frac{n}{2}+1],\\ \max\big\{p_{Fuj,m}(n);\lceil s\rceil;2\big\}< & p & \leq 1+\frac{2}{n-2s} & \text{if}\quad s\in(1,\frac{n}{2}).\end{array}\tag{2.56}$$

Then, there exists a small constant ϵ_0 such that if

$$\|(u_0,u_1)\|_{\mathcal{A}_{m,s,\varepsilon}}+\|(v_0,v_1)\|_{\mathcal{A}_{m,s,\varepsilon}}\leq\epsilon_0,$$

then there exists a uniquely determined globally (in time) energy solution to (2.3) in

$$\big(\mathcal{C}\big([0,\infty),H^s(\mathbb{R}^n)\big)\cap\mathcal{C}^1\big([0,\infty),H^{s-1+\varepsilon}(\mathbb{R}^n)\big)\big)^2.$$

Furthermore, the solution satisfies the following estimates:

$$\begin{aligned}\|(u,v)(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \big(1+B(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)}\\ &\qquad\times\big(\|(u_0,u_1)\|_{\mathcal{A}_{m,s,\varepsilon}}+\|(v_0,v_1)\|_{\mathcal{A}_{m,s,\varepsilon}}\big),\\ \|(u_t,v_t)(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim b(t)^{-1}\big(1+B(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-1}\\ &\qquad\times\big(\|(u_0,u_1)\|_{\mathcal{A}_{m,s,\varepsilon}}+\|(v_0,v_1)\|_{\mathcal{A}_{m,s,\varepsilon}}\big),\\ \|(|D|^s u,|D|^s v)(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \big(1+B(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s}{2}}\\ &\qquad\times\big(\|(u_0,u_1)\|_{\mathcal{A}_{m,s,\varepsilon}}+\|(v_0,v_1)\|_{\mathcal{A}_{m,s,\varepsilon}}\big),\\ \|(|D|^{s-1}u_t,|D|^{s-1}v_t)(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim b(t)^{-1}\big(1+B(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-1-\frac{s-1}{2}}\\ &\qquad\times\big(\|(u_0,u_1)\|_{\mathcal{A}_{m,s,\varepsilon}}+\|(v_0,v_1)\|_{\mathcal{A}_{m,s,\varepsilon}}\big).\end{aligned}$$

Remark 2.12. *If we have $p\neq q$, then we obtain the following result which is similar to Theorem 2.13.*

Theorem 2.14. *Let $n \geq 3$ and $s \in (1,3)$ be a non-integer number. The data (u_0, u_1) and (v_0, v_1) are supposed to belong to $\mathcal{A}_{m,s,\varepsilon} \times \mathcal{A}_{m,s,\varepsilon}$ with $m \in [1,2)$. Finally, the following conditions are satisfied for the exponents p and q:*

$$\begin{array}{llll} \max\left\{p_{Fuj,m}(n); \lceil s \rceil; 2\right\} < & \min\{p;q\} < \infty & & \text{if} \quad s \in [\frac{n}{2}, \frac{n}{2}+1], \\ \max\left\{p_{Fuj,m}(n); \lceil s \rceil; 2\right\} < & \min\{p;q\} < \max\{p;q\} & \leq 1 + \frac{2}{n-2s} & \text{if} \quad s \in (1, \frac{n}{2}). \end{array}$$

Then, there exists a small constant ϵ_0 such that if

$$\|(u_0,u_1)\|_{\mathcal{A}_{m,s,\varepsilon}} + \|(v_0,v_1)\|_{\mathcal{A}_{m,s,\varepsilon}} \leq \epsilon_0,$$

then there exists a uniquely determined globally (in time) energy solution to (2.1) in

$$\left(\mathcal{C}\big([0,\infty), H^s(\mathbb{R}^n)\big) \cap \mathcal{C}^1\big([0,\infty), H^{s-1+\varepsilon}(\mathbb{R}^n)\big)\right)^2.$$

Furthermore, the solution satisfies the same estimates as in Theorem 2.13.

2.3.2. Different orders of power nonlinearities and different regularities of the data

Now, we suppose that $\lceil s_1 \rceil \neq \lceil s_2 \rceil$ and that we have different exponents in the power nonlinearities. Then we may prove the following result.

Theorem 2.15. *Let $n \geq 4$. The regularity parameters s_1 and s_2 satisfy the following conditions:*

$$s_1, s_2 \in \left[3, \frac{n}{2}+1\right], \;\; 0 < s_2 - s_1 < 1 \;\; \text{and} \;\; \lceil s_1 \rceil \neq \lceil s_2 \rceil.$$

The data (u_0, u_1) and (v_0, v_1) are supposed to belong to $\mathcal{A}_{m,s_1} \times \mathcal{A}_{m,s_2}$ with $m \in [1,2)$. Furthermore, we assume for the exponents p and q the following conditions:

$$\begin{array}{llll} \lceil s_1 \rceil < p, & \lceil s_2 \rceil < q & \text{if} & n \leq 2s_1, \\ \lceil s_1 \rceil < p, & \lceil s_2 \rceil < q \leq 1 + \frac{2}{n-2s_1} & \text{if} & 2s_1 < n \leq 2s_2, \\ \lceil s_1 \rceil < p \leq 1 + \frac{2}{n-2s_2}, & \lceil s_2 \rceil < q \leq 1 + \frac{2}{n-2s_1} & \text{if} & n > 2s_2. \end{array} \tag{2.57}$$

Then, there exists a uniquely determined globally (in time) energy solution to (2.1) in

$$\begin{aligned} &\big(\mathcal{C}([0,\infty), H^{s_1}(\mathbb{R}^n)) \cap \mathcal{C}^1([0,\infty), H^{s_1-1}(\mathbb{R}^n))\big) \\ &\quad \times \big(\mathcal{C}([0,\infty), H^{s_2}(\mathbb{R}^n)) \cap \mathcal{C}^1([0,\infty), H^{s_2-1}(\mathbb{R}^n))\big). \end{aligned}$$

Furthermore, the solution satisfies the following estimates:

$$\begin{aligned}
\|(u,v)(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \big(1+B(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)}\left(\|(u_0,u_1)\|_{\mathcal{A}_{m,s_1}}+\|(v_0,v_1)\|_{\mathcal{A}_{m,s_2}}\right),\\
\|(u_t,v_t)(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim b(t)^{-1}\big(1+B(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-1}\\
&\qquad\times\left(\|(u_0,u_1)\|_{\mathcal{A}_{m,s_1}}+\|(v_0,v_1)\|_{\mathcal{A}_{m,s_2}}\right),\\
\||D|^{s_1}u(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \big(1+B(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s_1}{2}}\left(\|(u_0,u_1)\|_{\mathcal{A}_{m,s_1}}+\|(v_0,v_1)\|_{\mathcal{A}_{m,s_2}}\right),\\
\||D|^{s_1-1}u_t(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim b(t)^{-1}\big(1+B(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-1-\frac{s_1-1}{2}}\\
&\qquad\times\left(\|(u_0,u_1)\|_{\mathcal{A}_{m,s_1}}+\|(v_0,v_1)\|_{\mathcal{A}_{m,s_2}}\right),\\
\||D|^{s_2}v(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \big(1+B(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s_2}{2}}\left(\|(u_0,u_1)\|_{\mathcal{A}_{m,s_1}}+\|(v_0,v_1)\|_{\mathcal{A}_{m,s_2}}\right),\\
\||D|^{s_2-1}v_t(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim b(t)^{-1}\big(1+B(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-1-\frac{s_2-1}{2}}\\
&\qquad\times\left(\|(u_0,u_1)\|_{\mathcal{A}_{m,s_1}}+\|(v_0,v_1)\|_{\mathcal{A}_{m,s_2}}\right).
\end{aligned}$$

Example 2.8. *Let us choose $n=10$. If we are interested to get a larger admissible range, then we can choose for example $s_1=5, s_2=5+\frac{1}{10}$ to obtain $p>\lceil s_1\rceil=5$, $q>\lceil s_2\rceil=6$. We remark that a small negative perturbation of s_1 for example $s_1=5-\frac{1}{10}$ generates a restriction to the admissible range for q, this means, $p>\lceil s_1\rceil=5$ and $\lceil s_2\rceil=6<q\leq 11$. But, if we have fixed $p=5+\frac{1}{10}$ and $q=6+\frac{1}{10}$, then we can choose $s_1=5-\frac{1}{10}$ and $s_2=5+\frac{1}{10}$.*

Proof. We define the solution space $X(t)$ by

$$\begin{aligned}
X(t) &= \big\{(u,v)\in\big[\mathcal{C}([0,t],H^{s_1}(\mathbb{R}^n))\cap\mathcal{C}^1([0,t],H^{s_1-1}(\mathbb{R}^n))\big]\\
&\qquad\times\big[\mathcal{C}([0,t],H^{s_2}(\mathbb{R}^n))\cap\mathcal{C}^1([0,t],H^{s_2-1}(\mathbb{R}^n))\big]\big\}
\end{aligned}$$

with the norm

$$\|(u,v)\|_{X(t)}=\sup_{\tau\in[0,t]}\big\{M(\tau,u)+M(\tau,v)\big\},$$

where

$$\begin{aligned}
M(\tau,u) &= \big(1+B(\tau,0)\big)^{\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)}\|u(\tau,\cdot)\|_{L^2(\mathbb{R}^n)}\\
&\quad+b(\tau)\big(1+B(\tau,0)\big)^{\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)+1}\|u_t(\tau,\cdot)\|_{L^2(\mathbb{R}^n)}\\
&\quad+b(\tau)\big(1+B(\tau,0)\big)^{\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)+\frac{s_1-1}{2}+1}\||D|^{s_1-1}u_t(\tau,\cdot)\|_{L^2(\mathbb{R}^n)}\\
&\quad+\big(1+B(\tau,0)\big)^{\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)+\frac{s_1}{2}}\||D|^{s_1}u(\tau,\cdot)\|_{L^2(\mathbb{R}^n)},
\end{aligned}$$

and

$$\begin{aligned}
M(\tau,v) &= \big(1+B(\tau,0)\big)^{\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)}\|v(\tau,\cdot)\|_{L^2(\mathbb{R}^n)}\\
&\quad+b(\tau)\big(1+B(\tau,0)\big)^{\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)+1}\|v_t(\tau,\cdot)\|_{L^2(\mathbb{R}^n)}\\
&\quad+b(\tau)\big(1+B(\tau,0)\big)^{\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)+\frac{s_2-1}{2}+1}\||D|^{s_2-1}v_t(\tau,\cdot)\|_{L^2(\mathbb{R}^n)}\\
&\quad+\big(1+B(\tau,0)\big)^{\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)+\frac{s_2}{2}}\||D|^{s_2}v(\tau,\cdot)\|_{L^2(\mathbb{R}^n)}.
\end{aligned}$$

Let N be the mapping as defined in (2.11). Then our aim is to prove the following inequalities:

$$\|N(u,v)\|_{X(t)} \lesssim \|(u_0,u_1)\|_{\mathcal{A}_{m,s_1}} + \|(v_0,v_1)\|_{\mathcal{A}_{m,s_2}} + \|(u,v)\|^p_{X(t)} + \|(u,v)\|^q_{X(t)}, \quad (2.58)$$

$$\begin{aligned} \|N(u,v)-N(\tilde{u},\tilde{v})\|_{X(t)} &\lesssim \|(u,v)-(\tilde{u},\tilde{v})\|_{X(t)} \\ &\times\big(\|(u,v)\|^{p-1}_{X(t)} + \|(\tilde{u},\tilde{v})\|^{p-1}_{X(t)} + \|(u,v)\|^{q-1}_{X(t)} + \|(\tilde{u},\tilde{v})\|^{q-1}_{X(t)}\big). \end{aligned} \quad (2.59)$$

We can immediately conclude the desired estimate for the "linear part" of $N(u,v)$. It holds

$$\|(u^{ln},v^{ln})\|_{X(t)} \lesssim \|(u_0,u_1)\|_{\mathcal{A}_{m,s_1}} + \|(v_0,v_1)\|_{\mathcal{A}_{m,s_2}}. \quad (2.60)$$

For the "nonlinear terms" appearing in $N(u,v)$ we proceed as follows: From the estimate (1.27) of Theorem 1.6, with $m=2$ for τ from the integral over $[\frac{t}{2},t]$, we obtain for u^{nl} the estimate

$$\begin{aligned} &\||D|^{s_1-1}u^{nl}_t(t,\cdot)\|_{L^2(\mathbb{R}^n)} \\ &\lesssim \int_0^{\frac{t}{2}} b(t)^{-1}b(\tau)^{-1}(1+B(t,\tau))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s_1-1}{2}-1}\||v(\tau,x)|^p\|_{L^m(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)\cap \dot{H}^{s_1-1}(\mathbb{R}^n)}d\tau \\ &\quad+\int_{\frac{t}{2}}^t b(t)^{-1}b(\tau)^{-1}(1+B(t,\tau))^{-1-\frac{s_1-1}{2}}\||v(\tau,x)|^p\|_{L^2(\mathbb{R}^n)\cap \dot{H}^{s_1-1}(\mathbb{R}^n)}d\tau. \end{aligned}$$

Now we estimate successively $\||v(\tau,x)|^p\|_{L^2(\mathbb{R}^n)}$, $\||v(\tau,x)|^p\|_{L^m(\mathbb{R}^n)}$ and $\||v(\tau,x)|^p\|_{\dot{H}^{s_1-1}(\mathbb{R}^n)}$. Using Gagliardo-Nirenberg inequality for $0\le\tau\le t$ we obtain

$$\|v(\tau,\cdot)\|^p_{L^{mp}(\mathbb{R}^n)} \lesssim (1+B(\tau,0))^{-\frac{n}{2m}p+\frac{n}{2m}}\|(u,v)\|^p_{X(t)},$$

where

$$\begin{array}{ll} \frac{2}{m}\le p & \text{if } n\le 2s_2, \\ \frac{2}{m}\le p\le \frac{2n}{m(n-2s_2)} & \text{if } n>2s_2. \end{array}$$

The same ideas lead for $0\le\tau\le t$ to the estimate

$$\|v(\tau,\cdot)\|^p_{L^{2p}(\mathbb{R}^n)} \lesssim (1+B(\tau,0))^{-\frac{n}{2m}p+\frac{n}{4}}\|(u,v)\|^p_{X(t)},$$

where

$$\begin{array}{ll} \frac{2}{m}\le p & \text{if } n\le 2s_2, \\ \frac{2}{m}\le p\le \frac{n}{n-2s_2} & \text{for } n>2s_2. \end{array}$$

All together implies the following conditions for the exponent p:

$$\begin{array}{ll} \frac{2}{m}\le p & \text{if } n\le 2s_2, \\ \frac{2}{m}\le p\le \frac{n}{n-2s_2} & \text{if } n>2s_2. \end{array}$$

These condition are included in (2.57).
To estimate $\||v(\tau,x)|^p\|_{\dot{H}^{s_1-1}(\mathbb{R}^n)}$ we use the fractional chain rule from Proposition A.4. In this way it follows

$$\begin{aligned} \||v(\tau,x)|^p\|_{\dot{H}^{s_1-1}(\mathbb{R}^n)} &= \big\||D|^{s_1-1}|v(\tau,x)|^p\big\|_{L^2(\mathbb{R}^n)} \\ &\lesssim \big\|v(\tau,\cdot)\big\|^{p-1}_{L^{q_1}(\mathbb{R}^n)}\big\||D|^{s_1-1}v(\tau,\cdot)\big\|_{L^{q_2}(\mathbb{R}^n)} \text{ for } p>\lceil s_1-1\rceil, \end{aligned}$$

where
$$\frac{p-1}{q_1}+\frac{1}{q_2}=\frac{1}{2}.$$
Applying the classical Gagliardo-Nirenberg inequality to $\|v(\tau,\cdot)\|_{L^{q_1}(\mathbb{R}^n)}$ we get

$$\begin{aligned}\big\|v(\tau,\cdot)\big\|_{L^{q_1}(\mathbb{R}^n)} &\lesssim \big\|v(\tau,\cdot)\big\|^{1-\theta}_{L^2(\mathbb{R}^n)}\big\||D|^{s_2}v(\tau,\cdot)\big\|^{\theta}_{L^2(\mathbb{R}^n)}\\ &\lesssim (1+B(\tau,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)(1-\theta)-(\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)+\frac{s_2}{2})\theta}\|(u,v)\|_{X(t)}\\ &\lesssim (1+B(\tau,0))^{-\frac{n}{2m}+\frac{n}{2q_1}}\|(u,v)\|_{X(t)},\end{aligned}$$

where $\theta=\frac{n}{s_2}\big(\frac{1}{2}-\frac{1}{q_1}\big)\in[0,1]$. This implies

$$\begin{array}{ll} 2\leq q_1 & \text{if} \;\; n\leq 2s_2,\\ 2\leq q_1\leq \frac{n}{n-2s_2} & \text{if} \;\; n>2s_2.\end{array}$$

Finally, for $0\leq\tau\leq t$ we arrive at

$$\big\|v(\tau,\cdot)\big\|^{p-1}_{L^{q_1}(\mathbb{R}^n)}\lesssim (1+B(\tau,0))^{\left(-\frac{n}{2m}+\frac{n}{2q_1}\right)(p-1)}\|(u,v)\|^{p-1}_{X(t)}. \tag{2.61}$$

We apply the same tools to estimate $\big\||D|^{s_1-1}v(\tau,\cdot)\big\|_{L^{q_2}(\mathbb{R}^n)}$. So, if $s_1-1<s_2$, then we obtain

$$\begin{aligned}\big\||D|^{s_1-1}v(\tau,\cdot)\big\|_{L^{q_2}(\mathbb{R}^n)} &\lesssim \big\|v(\tau,\cdot)\big\|^{1-\tilde\theta}_{L^2(\mathbb{R}^n)}\big\||D|^{s_2}v(\tau,\cdot)\big\|^{\tilde\theta}_{L^2(\mathbb{R}^n)}\\ &\lesssim (1+B(\tau,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)(1-\tilde\theta)-\left(\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)+\frac{s_2}{2}\right)\tilde\theta}\|(u,v)\|_{X(t)}\\ &\lesssim (1+B(\tau,0))^{-\frac{n}{2m}+\frac{n}{2q_2}-\frac{s_2-1}{2}}\|(u,v)\|_{X(t)},\end{aligned}$$

where $\tilde\theta=\frac{n}{s_2}\big(\frac{1}{2}-\frac{1}{q_2}\big)+\frac{s_2-1}{s_2}\in\big[\frac{s_2-1}{s_2},1\big]$. This implies

$$\begin{array}{ll} 2\leq q_2 & \text{if} \;\; n\leq 2,\\ 2\leq q_2\leq \frac{2n}{n-2} & \text{if} \;\; n>2.\end{array}$$

The existence of such q_1,q_2 satisfying the above conditions was explained in Chapter 1, Section 1.4 (see the proof of Theorem 1.10).
Consequently, we get for $0\leq\tau\leq t$ the estimate

$$\||v(\tau,x)|^p\|_{\dot H^{s_1-1}(\mathbb{R}^n)}\lesssim (1+B(\tau,0))^{-\frac{n}{2m}p+\frac{n}{4}-\frac{s_2-1}{2}}\|(u,v)\|^p_{X(t)}$$

for $p>\lceil s_1-1\rceil$.
Using the last estimates we obtain for $\tau\in[0,\frac{t}{2}]$ the estimate

$$\begin{aligned}&\int_0^{\frac{t}{2}} b(t)^{-1}b(\tau)^{-1}(1+B(t,\tau))^{-\frac{n}{4}-\frac{s_1-1}{2}-1}(1+B(\tau,0))^{-\frac{n}{2m}p+\frac{n}{2m}}d\tau\\ &\quad\lesssim b(t)^{-1}(1+B(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s_1-1}{2}-1}\int_0^{\frac{t}{2}} b(\tau)^{-1}(1+B(\tau,0))^{-\frac{n}{2m}p+\frac{n}{2m}}d\tau\\ &\quad\lesssim b(t)^{-1}(1+B(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s_1-1}{2}-1},\end{aligned}$$

where we use the assumption $p > p_{Fuj,m}(n)$.
If $\tau \in [\frac{t}{2}, t]$, then we have

$$\begin{aligned}&\int_{\frac{t}{2}}^{t} b(t)^{-1}b(\tau)^{-1}(1+B(t,\tau))^{-1-\frac{s_1-1}{2}}(1+B(\tau,0))^{-\frac{n}{2m}p+\frac{n}{4}}d\tau\\
&\quad\lesssim b(t)^{-1}(1+B(t,0))^{-\frac{n}{2m}p+\frac{n}{4}}\\
&\quad\lesssim b(t)^{-1}(1+B(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s_1-1}{2}-1}\end{aligned}$$

for $p > \frac{2m}{n}\left(\frac{s_1+1}{2}\right)+1$ which follows from $p > s_1$ if we assume $n \geq 4$ and $s_1 \geq 3$. Consequently, we get

$$\||D|^{s_1-1}u_t^{nl}(t,\cdot)\|_{L^2(\mathbb{R}^n)} \lesssim b(t)^{-1}(1+B(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s_1-1}{2}-1}\|(u,v)\|^p_{X(t)}.$$

In the same way we can prove

$$\begin{aligned}\|u^{nl}(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim (1+B(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)}\|(u,v)\|^p_{X(t)},\\
\|u_t^{nl}(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim b(t)^{-1}(1+B(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-1}\|(u,v)\|^p_{X(t)},\\
\||D|^{s_1}u_t^{nl}(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim (1+B(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s_1}{2}}\|(u,v)\|^p_{X(t)}.\end{aligned}$$

Summarizing implies for u^{nl} the estimate

$$M\big(t,u^{nl}\big) \lesssim \|(u,v)\|^p_{X(t)}. \tag{2.62}$$

To treat v^{nl} we get after using the estimate (1.27) of Theorem 1.6, with $m=2$ for τ from the interval $[\frac{t}{2}, t]$,

$$\begin{aligned}&\||D|^{s_2-1}v_t^{nl}(t,\cdot)\|_{L^2(\mathbb{R}^n)}\\
&\lesssim \int_0^{\frac{t}{2}} b(t)^{-1}b(\tau)^{-1}(1+B(t,\tau))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s_2-1}{2}-1}\||u(\tau,\cdot)|^q\|_{L^m(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)\cap \dot{H}^{s_2-1}(\mathbb{R}^n)}d\tau\\
&\qquad+\int_{\frac{t}{2}}^{t} b(t)^{-1}b(\tau)^{-1}(1+B(t,\tau))^{-1-\frac{s_2-1}{2}}\||u(\tau,\cdot)|^q\|_{L^2(\mathbb{R}^n)\cap \dot{H}^{s_2-1}(\mathbb{R}^n)}d\tau.\end{aligned}$$

Analogously to the treatment of u^{nl} we can estimate successively $\||u(\tau,\cdot)|^q\|_{L^2(\mathbb{R}^n)}$, $\||u(\tau,\cdot)|^q\|_{L^m(\mathbb{R}^n)}$ and $\||u(\tau,\cdot)|^q\|_{\dot{H}^{s_2-1}(\mathbb{R}^n)}$. In this way we obtain for $0 \leq \tau \leq t$ the estimates

$$\begin{aligned}\||u(\tau,\cdot)|^q\|_{L^m(\mathbb{R}^n)} &\lesssim (1+B(\tau,0))^{-\frac{n}{2m}q+\frac{n}{2m}}\|(u,v)\|^q_{X(t)},\\
\||u(\tau,\cdot)|^q\|_{L^2(\mathbb{R}^n)} &\lesssim (1+B(\tau,0))^{-\frac{n}{2m}q+\frac{n}{4}}\|(u,v)\|^q_{X(t)},\\
\||u(\tau,\cdot)|^q\|_{\dot{H}^{s_2-1}(\mathbb{R}^n)} &\lesssim (1+B(\tau,0))^{-\frac{n}{2m}q+\frac{n}{4}-\frac{s_2-1}{2}}\|(u,v)\|^q_{X(t)},\end{aligned}$$

where

$$\begin{aligned}&\tfrac{2}{m} \leq q && \text{if } n \leq 2s_1,\\
&\tfrac{2}{m} \leq q \leq 1+\tfrac{2}{n-2s_1} && \text{if } n > 2s_1,\end{aligned}$$

and $q > \lceil s_2 - 1 \rceil, s_2 - 1 < s_1$.
Using the last estimates we obtain for $\tau \in [0, \frac{t}{2}]$

$$
\begin{aligned}
&\int_0^{\frac{t}{2}} b(t)^{-1} b(\tau)^{-1} (1 + B(t,\tau))^{-\frac{n}{4} - \frac{s_2-1}{2} - 1} (1 + B(\tau, 0))^{-\frac{n}{2m}q + \frac{n}{2m}} d\tau \\
&\quad \lesssim b(t)^{-1} (1 + B(t,0))^{-\frac{n}{2}\left(\frac{1}{m} - \frac{1}{2}\right) - \frac{s_2-1}{2} - 1} \int_0^{\frac{t}{2}} b(\tau)^{-1} (1 + B(\tau,0))^{-\frac{n}{2m}q + \frac{n}{2m}} d\tau \\
&\quad \lesssim b(t)^{-1} (1 + B(t,0))^{-\frac{n}{2}\left(\frac{1}{m} - \frac{1}{2}\right) - \frac{s_2-1}{2} - 1},
\end{aligned}
$$

where we used the assumption $q > p_{Fuj,m}(n)$.
If $\tau \in [\frac{t}{2}, t]$, then we have

$$
\begin{aligned}
&\int_{\frac{t}{2}}^{t} b(t)^{-1} b(\tau)^{-1} (1 + B(t,\tau))^{-1 - \frac{s_2-1}{2}} (1 + B(\tau, 0))^{-\frac{n}{2m}q + \frac{n}{4}} d\tau \\
&\quad \lesssim b(t)^{-1} (1 + B(t,0))^{-\frac{n}{2m}q + \frac{n}{4}} \\
&\quad \lesssim b(t)^{-1} (1 + B(t,0))^{-\frac{n}{2}\left(\frac{1}{m} - \frac{1}{2}\right) - \frac{s_2-1}{2} - 1}
\end{aligned}
$$

for $q > \frac{2m}{n}\left(\frac{s_2+1}{2}\right) + 1$ which follows from the condition $q > s_2$ in (2.57) and the other assumptions of the theorem.
Consequently, we get

$$
\| |D|^{s_2-1} v_t^{nl}(t,\cdot)\|_{L^2(\mathbb{R}^n)} \lesssim b(t)^{-1} (1 + B(t,0))^{-\frac{n}{2}\left(\frac{1}{m} - \frac{1}{2}\right) - \frac{s_2-1}{2} - 1} \|(u,v)\|_{X(t)}^q.
$$

In the same way we can prove

$$
\begin{aligned}
\|v^{nl}(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim (1 + B(t,0))^{-\frac{n}{2}\left(\frac{1}{m} - \frac{1}{2}\right)} \|(u,v)\|_{X(t)}^q, \\
\|v_t^{nl}(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim b(t)^{-1} (1 + B(t,0))^{-\frac{n}{2}\left(\frac{1}{m} - \frac{1}{2}\right) - 1} \|(u,v)\|_{X(t)}^q, \\
\| |D|^{s_2} v^{nl}(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim (1 + B(t,0))^{-\frac{n}{2}\left(\frac{1}{m} - \frac{1}{2}\right) - \frac{s_2}{2}} \|(u,v)\|_{X(t)}^q.
\end{aligned}
$$

Summarizing, we may conclude for v^{nl} the following estimate:

$$
M\big(t, v^{nl}\big) \lesssim \|(u,v)\|_{X(t)}^q. \tag{2.63}
$$

From (2.62) and (2.63) we get (2.58).
Now for the uniqueness part we prove (2.59). We assume that (u,v) and $(\tilde{u},\tilde{v})$ are two vector-functions belonging to the space $X(t)$. Then we have

$$
\begin{aligned}
&N(u,v) - N(\tilde{u},\tilde{v}) \\
&\quad = \big(u^{nl}(t,x) - \tilde{u}^{nl}(t,x), v^{nl}(t,x) - \tilde{v}^{nl}(t,x)\big) \\
&\quad = \Big(\int_0^t E_1(t,\tau,x) *_{(x)} \big(|v(\tau,x)|^p - |\tilde{v}(\tau,x)|^p\big) d\tau, \\
&\qquad\quad \int_0^t E_1(t,\tau,x) *_{(x)} \big(|u(\tau,x)|^q - |\tilde{u}(\tau,x)|^q\big) d\tau \Big).
\end{aligned}
$$

To estimate the first component we use the estimates (1.27) from Theorem 1.6 and obtain

$$\begin{aligned}
&\Big\||D|^{s_1-1}\partial_t\int_0^t E_1(t,\tau,x)*_{(x)}\big(|v(\tau,x)|^p-|\tilde v(\tau,x)|^p\big)d\tau\Big\|_{L^2(\mathbb{R}^n)}\\
&\lesssim\int_0^{\frac t2} b(t)^{-1}b(\tau)^{-1}(1+B(t,\tau))^{-\frac n2\left(\frac1m-\frac12\right)-\frac{s_1-1}{2}-1}\\
&\qquad\qquad\times\||v(\tau,x)|^p-|\tilde v(\tau,x)|^p\|_{L^m(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)\cap\dot H^{s_1-1}(\mathbb{R}^n)}d\tau\\
&\quad+\int_{\frac t2}^t b(t)^{-1}b(\tau)^{-1}(1+B(t,\tau))^{-1-\frac{s_1-1}{2}}\\
&\qquad\qquad\times\||v(\tau,x)|^p-|\tilde v(\tau,x)|^p\|_{L^2(\mathbb{R}^n)\cap\dot H^{s_1-1}(\mathbb{R}^n)}d\tau.
\end{aligned}$$

Using Hölder's inequality and the Gagliardo-Nirenberg inequality we get

$$\begin{aligned}
\Big\||v(\tau,x)|^p-|\tilde v(\tau,x)|^p\Big\|_{L^m(\mathbb{R}^n)} &\lesssim (1+B(\tau,0))^{-\frac{n}{2m}p+\frac{n}{2m}}\sup_{\tau\in[0,t]}M(\tau,v-\tilde v)\\
&\qquad\times\big(\|(u,v)\|_{X(t)}^{p-1}+\|(\tilde u,\tilde v)\|_{X(t)}^{p-1}\big),
\end{aligned}$$

and

$$\begin{aligned}
\Big\||v(\tau,x)|^p-|\tilde v(\tau,x)|^p\Big\|_{L^2(\mathbb{R}^n)} &\lesssim (1+B(\tau,0))^{-\frac{n}{2m}p+\frac{n}{4}}\sup_{\tau\in[0,t]}M(\tau,v-\tilde v)\\
&\qquad\times\big(\|(u,v)\|_{X(t)}^{p-1}+\|(\tilde u,\tilde v)\|_{X(t)}^{p-1}\big).
\end{aligned}$$

To estimate $\||v(\tau,x)|^p-|\tilde v(\tau,x)|^p\|_{\dot H^{s_1-1}(\mathbb{R}^n)}$ we recall the estimates from the proof of Theorem 1.10. Applying these estimates we may conclude for $0\le\tau\le t$

$$\begin{aligned}
\Big\||v(\tau,x)|^p-|\tilde v(\tau,x)|^p\Big\|_{\dot H^{s_1-1}(\mathbb{R}^n)} &\lesssim (1+B(\tau,0))^{-\frac{n}{2m}p+\frac{n}{4}-\frac{s_2-1}{2}}\sup_{\tau\in[0,t]}M(\tau,v-\tilde v)\\
&\qquad\times\big(\|(u,v)\|_{X(t)}^{p-1}+\|(\tilde u,\tilde v)\|_{X(t)}^{p-1}\big),
\end{aligned}$$

where we use $p>\lceil s_1\rceil$ and $p\le 1+\frac{2}{n-2s_2}$ if $n>2s_2$.
All these estimates together imply

$$\begin{aligned}
&\Big\||D|^{s_1-1}\partial_t\int_0^t E_1(t,\tau,x)*_{(x)}\big(|v(\tau,x)|^p-|\tilde v(\tau,x)|^p\big)d\tau\Big\|_{L^2(\mathbb{R}^n)}\\
&\lesssim b(t)^{-1}(1+B(t,0))^{-\frac n2\left(\frac1m-\frac12\right)-\frac{s_1-1}{2}-1}\\
&\qquad\times\sup_{\tau\in[0,t]}M(\tau,v-\tilde v)\big(\|(u,v)\|_{X(t)}^{p-1}+\|(\tilde u,\tilde v)\|_{X(t)}^{p-1}\big).
\end{aligned}$$

In the same way we get

$$\begin{aligned}
&\Big\|\int_0^t E_1(t,\tau,x)*_{(x)}\big(|v(\tau,x)|^p-|\tilde v(\tau,x)|^p\big)d\tau\Big\|_{L^2(\mathbb{R}^n)}\\
&\lesssim(1+B(t,0))^{-\frac n2\left(\frac1m-\frac12\right)}\sup_{\tau\in[0,t]}M(\tau,v-\tilde v)\big(\|(u,v)\|_{X(t)}^{p-1}+\|(\tilde u,\tilde v)\|_{X(t)}^{p-1}\big),
\end{aligned}$$

$$\begin{aligned}
&\Big\|\partial_t\int_0^t E_1(t,\tau,x)*_{(x)}\big(|v(\tau,x)|^p-|\tilde v(\tau,x)|^p\big)d\tau\Big\|_{L^2(\mathbb{R}^n)}\\
&\lesssim b(t)^{-1}(1+B(t,0))^{-\frac n2\left(\frac1m-\frac12\right)-1}\sup_{\tau\in[0,t]}M(\tau,v-\tilde v)\big(\|(u,v)\|_{X(t)}^{p-1}+\|(\tilde u,\tilde v)\|_{X(t)}^{p-1}\big),
\end{aligned}$$

$$\begin{aligned}&\Big\||D|^{s_1}\int_0^t E_1(t,\tau,x)*_{(x)}\big(|v(\tau,x)|^p-|\tilde{v}(\tau,x)|^p\big)d\tau\Big\|_{L^2(\mathbb{R}^n)}\\&\qquad\lesssim(1+B(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s_1}{2}}\sup_{\tau\in[0,t]}M(\tau,v-\tilde{v})\big(\|(u,v)\|^{p-1}_{X(t)}+\|(\tilde{u},\tilde{v})\|^{p-1}_{X(t)}\big).\end{aligned}$$

For the second component of $N(u,v)-N(\tilde{u},\tilde{v})$ we get for $q>\lceil s_2\rceil$ and $q\leq 1+\frac{2}{n-2s_1}$ if $n>2s_1$, after applying the same ideas as above, the following estimates:

$$\begin{aligned}&\Big\|\int_0^t E_1(t,\tau,x)*_{(x)}\big(|u(\tau,x)|^q-|\tilde{u}(\tau,x)|^q\big)d\tau\Big\|_{L^2(\mathbb{R}^n)}\\&\qquad\lesssim(1+B(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)}\sup_{\tau\in[0,t]}M(\tau,u-\tilde{u})\big(\|(u,v)\|^{q-1}_{X(t)}+\|(\tilde{u},\tilde{v})\|^{q-1}_{X(t)}\big),\end{aligned}$$

$$\begin{aligned}&\Big\|\partial_t\int_0^t E_1(t,\tau,x)*_{(x)}\big(|u(\tau,x)|^q-|\tilde{u}(\tau,x)|^q\big)d\tau\Big\|_{L^2(\mathbb{R}^n)}\\&\qquad\lesssim b(t)^{-1}(1+B(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-1}\sup_{\tau\in[0,t]}M(\tau,u-\tilde{u})\big(\|(u,v)\|^{q-1}_{X(t)}+\|(\tilde{u},\tilde{v})\|^{q-1}_{X(t)}\big),\end{aligned}$$

$$\begin{aligned}&\Big\||D|^{s_2-1}\partial_t\int_0^t E_1(t,\tau,x)*_{(x)}\big(|u(\tau,x)|^q-|\tilde{u}(\tau,x)|^q\big)d\tau\Big\|_{L^2(\mathbb{R}^n)}\\&\qquad\lesssim b(t)^{-1}(1+B(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s_2-1}{2}-1}\\&\qquad\qquad\times\sup_{\tau\in[0,t]}M(\tau,u-\tilde{u})\big(\|(u,v)\|^{q-1}_{X(t)}+\|(\tilde{u},\tilde{v})\|^{q-1}_{X(t)}\big),\end{aligned}$$

$$\begin{aligned}&\Big\||D|^{s_2}\int_0^t E_1(t,\tau,x)*_{(x)}\big(|u(\tau,x)|^q-|\tilde{u}(\tau,x)|^q\big)d\tau\Big\|_{L^2(\mathbb{R}^n)}\\&\qquad\lesssim(1+B(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s_2}{2}}\sup_{\tau\in[0,t]}M(\tau,u-\tilde{u})\big(\|(u,v)\|^{q-1}_{X(t)}+\|(\tilde{u},\tilde{v})\|^{q-1}_{X(t)}\big).\end{aligned}$$

The proof is completed. □

Remark 2.13. *If we consider the opposite case to the case treated in Theorem 2.15, that is, $0<s_1-s_2<1$, then the condition (2.57) is modified as follows:*

$$\begin{array}{llll}\lceil s_1\rceil<p, & \lceil s_2\rceil<q & \text{if} & n\leq 2s_2,\\ \lceil s_1\rceil<p\leq 1+\frac{2}{n-2s_2}, & \lceil s_2\rceil<q & \text{if} & 2s_2<n\leq 2s_1,\\ \lceil s_1\rceil<p\leq 1+\frac{2}{n-2s_2}, & \lceil s_2\rceil<q\leq 1+\frac{2}{n-2s_1} & \text{if} & n>2s_1.\end{array}$$

2.4. Large regular data

Finally, we are interested in the case of data having a large regularity such that they belong to $L^\infty(\mathbb{R}^n)$, too. For this reason we choose the regularity parameters s_1 and s_2 from the interval $(\frac{n}{2}+1,\infty)$.

2.4.1. The regularity of data coincide

From the treatment of the case of large regular data in Chapter 1, Section 1.5 we recall that the admissible range for p was bounded to below by the regularity parameter s.

Theorem 2.16. *Let us assume $n \geq 4$. The data (u_0, u_1) and (v_0, v_1) are supposed to belong to $\mathcal{A}_{m,s} \times \mathcal{A}_{m,s}$ with $m \in [1, 2)$ and $s > \frac{n}{2} + 1$. Moreover, let*

$$p > s. \tag{2.64}$$

Then, there exists a small constant ϵ_0 such that if

$$\|(u_0, u_1)\|_{\mathcal{A}_{m,s}} + \|(v_0, v_1)\|_{\mathcal{A}_{m,s}} \leq \epsilon_0,$$

then there exists a uniquely determined globally (in time) energy solution to (2.3) in

$$\big(\mathcal{C}([0, \infty), H^s(\mathbb{R}^n)) \cap \mathcal{C}^1([0, \infty), H^{s-1}(\mathbb{R}^n))\big)^2 .$$

The solution satisfies the following estimates:

$$\begin{aligned}
\|(u, v)(t, \cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \big(1 + B(t, 0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)}\big(\|(u_0, u_1)\|_{\mathcal{A}_{m,s}} + \|(v_0, v_1)\|_{\mathcal{A}_{m,s}}\big),\\
\|(u_t, v_t)(t, \cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim b(t)^{-1}\big(1 + B(t, 0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-1}\\
&\qquad \times\big(\|(u_0, u_1)\|_{\mathcal{A}_{m,s}} + \|(v_0, v_1)\|_{\mathcal{A}_{m,s}}\big),\\
\||D|^s(u, v)(t, \cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \big(1 + B(t, 0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s}{2}}\\
&\qquad \times\big(\|(u_0, u_1)\|_{\mathcal{A}_{m,s}} + \|(v_0, v_1)\|_{\mathcal{A}_{m,s}}\big),\\
\||D|^{s-1}(u_t, v_t)(t, \cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim b(t)^{-1}\big(1 + B(t, 0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s-1}{2}-1}\\
&\qquad \times\big(\|(u_0, u_1)\|_{\mathcal{A}_{m,s}} + \|(v_0, v_1)\|_{\mathcal{A}_{m,s}}\big).
\end{aligned}$$

One can prove this theorem by following the same steps of the proof of Theorem 1.15.
If we have $p \neq q$ and both exponents satisfy the condition (2.64), then we may immediately conclude from the results of Chapter 1 the following theorem.

Theorem 2.17. *Let us assume $n \geq 4$. The data (u_0, u_1) and (v_0, v_1) are supposed to belong to $\mathcal{A}_{m,s} \times \mathcal{A}_{m,s}$ with $m \in [1, 2)$ and $s > \frac{n}{2} + 1$. Moreover, let*

$$\min\{p; q\} > s. \tag{2.65}$$

Then, there exists a small constant ϵ_0 such that if

$$\|(u_0, u_1)\|_{\mathcal{A}_{m,s}} + \|(v_0, v_1)\|_{\mathcal{A}_{m,s}} \leq \epsilon_0,$$

then there exists a uniquely determined globally (in time) energy solution to (2.1) in

$$\big(\mathcal{C}([0, \infty), H^s(\mathbb{R}^n)) \cap \mathcal{C}^1([0, \infty), H^{s-1}(\mathbb{R}^n))\big)^2 .$$

The solution satisfies the estimates of Theorem 2.16.

The condition (2.65) appears from the estimate of the norm $\||u(\tau,x)|^q\|_{\dot{H}^{s-1}(\mathbb{R}^n)}$ and $\||v(\tau,x)|^p\|_{\dot{H}^{s-1}(\mathbb{R}^n)}$ after using fractional powers for $s-1>\frac{n}{2}$ from Corollary A.3.

Remark 2.14. *Under the assumptions of Theorem 2.17, the use of fractional chain rule would imply more restrictive results. Precisely, the admissible ranges for the exponents p and q will be as follows:*

$$\min\{p;q\} > \lceil s \rceil.$$

2.4.2. Different exponents in the power nonlinearities and different regularity of the data

Now we are interested in the case of data having different regularity parameters and we have different exponents in the power nonlinearities.

Theorem 2.18. *Let $n\geq 4$. The data (u_0,u_1) and (v_0,v_1) are supposed to belong to $\mathcal{A}_{m,s_1}\times\mathcal{A}_{m,s_2}$ with $m\in[1,2)$ and $s_2>s_1>\frac{n}{2}+1$. Moreover, we assume*

$$p > s_1 \quad \text{and} \quad q > \tilde{s}_2, \tag{2.66}$$

where $\tilde{s}_2\in(s_1,s_1+1)$ and $\tilde{s}_2\leq s_2$. Then, there exists a small constant ϵ_0 such that if

$$\|(u_0,u_1)\|_{\mathcal{A}_{m,s_1}}+\|(v_0,v_1)\|_{\mathcal{A}_{m,s_2}}\leq\epsilon_0,$$

then there exists a uniquely determined globally (in time) energy solution to (2.1) in

$$\begin{aligned}&\big(\mathcal{C}([0,\infty),H^{s_1}(\mathbb{R}^n))\cap\mathcal{C}^1([0,\infty),H^{s_1-1}(\mathbb{R}^n))\big)\\ &\quad\times\big(\mathcal{C}([0,\infty),H^{\tilde{s_2}}(\mathbb{R}^n))\cap\mathcal{C}^1([0,\infty),H^{\tilde{s_2}-1}(\mathbb{R}^n))\big).\end{aligned}$$

Furthermore, the solution satisfies the following estimates:

$$\begin{aligned}
\|(u,v)(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \big(1+B(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)}\left(\|(u_0,u_1)\|_{\mathcal{A}_{m,s_1}}+\|(v_0,v_1)\|_{\mathcal{A}_{m,s_2}}\right),\\
\|(u_t,v_t)(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim b(t)^{-1}\big(1+B(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-1}\\
&\qquad\times\left(\|(u_0,u_1)\|_{\mathcal{A}_{m,s_1}}+\|(v_0,v_1)\|_{\mathcal{A}_{m,s_2}}\right),\\
\||D|^{s_1}u(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \big(1+B(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s_1}{2}}\\
&\qquad\times\left(\|(u_0,u_1)\|_{\mathcal{A}_{m,s_1}}+\|(v_0,v_1)\|_{\mathcal{A}_{m,s_2}}\right),\\
\||D|^{s_1-1}u_t(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim b(t)^{-1}\big(1+B(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-1-\frac{s_1-1}{2}}\\
&\qquad\times\left(\|(u_0,u_1)\|_{\mathcal{A}_{m,s_1}}+\|(v_0,v_1)\|_{\mathcal{A}_{m,s_2}}\right),\\
\||D|^{\tilde{s}_2}v(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \big(1+B(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{\tilde{s}_2}{2}}\left(\|(u_0,u_1)\|_{\mathcal{A}_{m,s_1}}+\|(v_0,v_1)\|_{\mathcal{A}_{m,s_2}}\right),\\
\||D|^{\tilde{s}_2-1}v_t(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim b(t)^{-1}\big(1+B(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-1-\frac{\tilde{s}_2-1}{2}}\\
&\qquad\times\left(\|(u_0,u_1)\|_{\mathcal{A}_{m,s_1}}+\|(v_0,v_1)\|_{\mathcal{A}_{m,s_2}}\right).
\end{aligned}$$

Example 2.9. *If $n=10$, $(u_0,u_1)\in\mathcal{A}_{\frac{3}{2},7}$ and $(v_0,v_1)\in\mathcal{A}_{\frac{3}{2},10}$, then there exists a global (in time) energy solution to (2.1) provided that $p>7$ and $q>s$ with $s\in[7,8]$. Consequently, we do not observe anymore the regularity parameter of the second data (v_0,v_1) in the regularity parameter of the solution.*

Proof. In the proof of this theorem we mix some tools from the previous proofs. On the one hand we use the same space of solution $X(t)$ and its norm from the proof of Theorem 2.15. On the other hand we use fractional powers rule from Corollary A.3 to prove the inequalities (2.58) and (2.59). We have immediately

$$\|(u^{ln}, v^{ln})\|_{X(t)} \lesssim \|(u_0, u_1)\|_{\mathcal{A}_{m,s_1}} + \|(v_0, v_1)\|_{\mathcal{A}_{m,s_2}}. \tag{2.67}$$

For the nonlinear terms, we conclude from the estimate (1.27) of Theorem 1.6 the following estimate:

$$\begin{aligned}
&\||D|^{s_1-1} u_t^{nl}(t,\cdot)\|_{L^2(\mathbb{R}^n)} \\
&\lesssim \int_0^{\frac{t}{2}} b(t)^{-1} b(\tau)^{-1}(1+B(t,\tau))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s_1-1}{2}-1} \||v(\tau,x)|^p\|_{L^m(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)\cap \dot{H}^{s_1-1}(\mathbb{R}^n)} d\tau \\
&\quad + \int_{\frac{t}{2}}^t b(t)^{-1} b(\tau)^{-1}(1+B(t,\tau))^{-1-\frac{s_1-1}{2}} \||v(\tau,x)|^p\|_{L^2(\mathbb{R}^n)\cap \dot{H}^{s_1-1}(\mathbb{R}^n)} d\tau.
\end{aligned}$$

Using Gagliardo-Nirenberg inequality for $0 \le \tau \le t$ we get

$$\begin{aligned}
\|v(\tau,\cdot)\|^p_{L^{mp}(\mathbb{R}^n)} &\lesssim (1+B(\tau,0))^{-\frac{n}{2m}p+\frac{n}{2m}} \|(u,v)\|^p_{X(t)}, \\
\|v(\tau,\cdot)\|^p_{L^{2p}(\mathbb{R}^n)} &\lesssim (1+B(\tau,0))^{-\frac{n}{2m}p+\frac{n}{4}} \|(u,v)\|^p_{X(t)},
\end{aligned}$$

where $p \ge \frac{2}{m}$. To estimate $\||v(\tau,x)|^p\|_{\dot{H}^{s_1-1}(\mathbb{R}^n)}$ we use the fractional powers rules from Corollary A.3 and Proposition A.6. From the definition of the norm of the solution space $X(t)$ we obtain for $s_1 - 1 \le s_2$ and $0 \le \tau \le t$ the estimate

$$\||v(\tau,x)|^p\|_{\dot{H}^{s_1-1}(\mathbb{R}^n)} \lesssim (1+B(\tau,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)p-\frac{s_1-1}{2}-\frac{s^*}{2}(p-1)} \|(u,v)\|^p_{X(t)},$$

where $s^* = \frac{n}{2} - \varepsilon < \frac{n}{2}$ and $p > s_1$.
Repeating the same steps to derive the estimate (1.98) from the proof of Theorem 1.15 it follows

$$\||D|^{s_1-1} u_t^{nl}(t,\cdot)\|_{L^2(\mathbb{R}^n)} \lesssim \|(u,v)\|^p_{X(t)} b(t)^{-1}(1+B(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-1-\frac{s_1-1}{2}}, \tag{2.68}$$

where we use $p > p_{Fuj,m}(n)$ which follows from (2.66) because we suppose $s_1 > \frac{n}{2}+1$ there.
In a similar way we get for $p > s_1$ the estimates

$$\begin{aligned}
\|u^{nl}(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim (1+B(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)} \|(u,v)\|^p_{X(t)}, \\
\|u_t^{nl}(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim b(t)^{-1}(1+B(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-1} \|(u,v)\|^p_{X(t)}, \\
\||D|^{s_1} u^{nl}(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim (1+B(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s_1}{2}} \|(u,v)\|^p_{X(t)}.
\end{aligned}$$

This implies

$$M(t, u^{nl}) \lesssim \|(u,v)\|^p_{X(t)}.$$

Analogously, we obtain

$$M(t, v^{nl}) \lesssim \|(u,v)\|^q_{X(t)}.$$

Summarizing, (2.58) is proved. To prove (2.59) we prove similar estimates to (1.108) for $0 \leq \tau \leq t$ which are

$$\begin{aligned}
\||v(\tau,x)|^p - |\tilde{v}(\tau,x)|^p\|_{\dot{H}^{s_1-1}(\mathbb{R}^n)} &\lesssim (1+B(\tau,0))^{-\frac{n}{2m}p+\frac{n}{4}(p-1)-\frac{s_1-1}{2}} \sup_{\tau\in[0,t]} M(\tau, v-\tilde{v}) \\
&\quad \times \left(\|(u,v)\|_{X(t)}^{p-1} + \|(\tilde{u},\tilde{v})\|_{X(t)}^{p-1}\right), \\
\||u(\tau,x)|^q - |\tilde{u}(\tau,x)|^q\|_{\dot{H}^{\tilde{s}_2-1}(\mathbb{R}^n)} &\lesssim (1+B(\tau,0))^{-\frac{n}{2m}q+\frac{n}{4}(q-1)-\frac{\tilde{s}_2-1}{2}} \sup_{\tau\in[0,t]} M(\tau, u-\tilde{u}) \\
&\quad \times \left(\|(u,v)\|_{X(t)}^{q-1} + \|(\tilde{u},\tilde{v})\|_{X(t)}^{q-1}\right).
\end{aligned}$$

The proof can be completed similarly as we did in the proof of Theorem 2.15. □

Remark 2.15. *If we consider the case $s_1 > s_2 > \frac{n}{2}+1$, then the condition (2.66) is modified to*

$$p > \tilde{s}_1 \quad \textit{and} \quad q > s_2,$$

where $\tilde{s}_1 \in (s_2, s_2+1)$ and $\tilde{s}_1 \leq s_1$. Then the loss of regularity will be with respect to the first component u of the solution (u,v). For this reason the solution belongs to

$$\begin{aligned}
&\big(\mathcal{C}([0,\infty), H^{\tilde{s}_1}(\mathbb{R}^n)) \cap \mathcal{C}^1([0,\infty), H^{\tilde{s}_1-1}(\mathbb{R}^n))\big) \\
&\quad \times \big(\mathcal{C}([0,\infty), H^{s_2}(\mathbb{R}^n)) \cap \mathcal{C}^1([0,\infty), H^{s_2-1}(\mathbb{R}^n))\big).
\end{aligned}$$

2.5. Concluding remarks

From the statements of the previous theorems we want to conclude some results similar as we did in the last section of Chapter 1. We collect some results describing the admissible range of exponents in both power nonlinearities with respect to the modified Fujita exponent or to the regularity parameters of both data.

Theorem 2.19. *Let $n \leq 2$, $s \in \left[\frac{n}{2}, n\right] \cap [0,1]$ and $m \in [1,2)$. Let us assume for the data the following condition:*

$$(u_0,u_1), (v_0,v_1) \in \mathcal{A}_{m,s} \times \mathcal{A}_{m,s}.$$

Finally, let

$$\min\{p;q\} > p_{Fuj,m}(n).$$

Then, there exists a small constant ϵ_0 such that if

$$\|(u_0,u_1)\|_{\mathcal{A}_{m,s}} + \|(v_0,v_1)\|_{\mathcal{A}_{m,s}} \leq \epsilon_0,$$

then there exists a uniquely determined globally (in time) Sobolev solution to (2.1) in

$$\big(\mathcal{C}\big([0,\infty), H^s(\mathbb{R}^n)\big) \cap \mathcal{C}^{\lfloor \min\{s;1\}\rfloor}\big([0,\infty), L^2(\mathbb{R}^n)\big)\big)^2.$$

Furthermore, the solution satisfies for $s_0 + l = 0, s$, where $s_0 \in [0,s]$, $l = 0,1$, the estimates

$$\begin{aligned}
\|(|D|^{s_0}\partial_t^l u, |D|^{s_0}\partial_t^l v)(t,\cdot)\|_{L^2(\mathbb{R}^n)} \quad &\lesssim \quad \big(1+B(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-l-\frac{s_0}{2}} (b(t))^{-l} \\
&\quad \times \left(\|(u_0,u_1)\|_{\mathcal{A}_{m,s}} + \|(v_0,v_1)\|_{\mathcal{A}_{m,s}}\right).
\end{aligned}$$

Theorem 2.20. *Let $n \geq 4$. Let us assume for the data the following condition:*

$$(u_0, u_1), (v_0, v_1) \in \mathcal{A}_{m,s_1} \times \mathcal{A}_{m,s_2}, s_1, s_2 \in \Big[\frac{n}{2}, \infty\Big), s_1 - s_2 \in [-1, 1].$$

Finally, let

$$\begin{aligned} p > \lceil s_1 \rceil \quad \text{and} \quad q > \lceil s_2 \rceil \quad \text{if} \quad s_1, s_2 \in \Big[\frac{n}{2}, \frac{n}{2} + 1\Big], \\ p > s_1 \quad \text{and} \quad q > s_2 \quad \text{if} \quad s_1, s_2 > \frac{n}{2} + 1. \end{aligned}$$

Then, there exists a small constant ϵ_0 such that if

$$\|(u_0, u_1)\|_{\mathcal{A}_{m,s_1}} + \|(v_0, v_1)\|_{\mathcal{A}_{m,s_2}} \leq \epsilon_0,$$

then there exists a uniquely determined globally (in time) energy solution to (2.1) belonging to

$$\begin{aligned} &\Big(\mathcal{C}\big([0,\infty), H^{s_1}(\mathbb{R}^n)\big) \cap \mathcal{C}^1\big([0,\infty), H^{s_1-1}(\mathbb{R}^n)\big)\Big) \\ &\quad \times \Big(\mathcal{C}\big([0,\infty), H^{s_2}(\mathbb{R}^n)\big) \cap \mathcal{C}^1\big([0,\infty), H^{s_2-1}(\mathbb{R}^n)\big)\Big). \end{aligned}$$

Furthermore, the solution satisfies for $s_0 + l = 0, s$, where $s_0 \in [0, s]$, $l = 0, 1$, the estimates

$$\begin{aligned} \|(|D|^{s_0}\partial_t^l u, |D|^{s_0}\partial_t^l v)(t,\cdot)\|_{L^2(\mathbb{R}^n)} \lesssim\ & \big(1 + B(t,0)\big)^{-\frac{n}{4} - l - \frac{s_0}{2}} b(t)^{-l} \\ & \times \left(\|(u_0, u_1)\|_{\mathcal{A}_{m,s_1}} + \|(v_0, v_1)\|_{\mathcal{A}_{m,s_2}}\right). \end{aligned}$$

3. Weakly coupled systems of semilinear damped waves with different coefficients in the dissipation terms

This chapter is devoted to the particular case of the system (0.2), where the source terms are $g(v) = |v|^p$ and $f(u) = |u|^q$ and the dissipation terms are given by $b_1(t)u_t$ and $b_2(t)v_t$ with $b_1(t) = \frac{1}{(1+t)^{r_1}}$ and $b_2(t) = \frac{1}{(1+t)^{r_2}}$ with exponents $r_1,\, r_2 \in (-1,1)$. For this reason the dissipation terms are effective in the sense of [67] and [69] (see Example 1.1). The model we have in mind is

$$\begin{aligned} &u_{tt} - \Delta u + \tfrac{1}{(1+t)^{r_1}} u_t = |v|^p, \qquad v_{tt} - \Delta v + \tfrac{1}{(1+t)^{r_2}} v_t = |u|^q,\\ &u(0,x) = u_0(x),\;\; u_t(0,x) = u_1(x),\;\; v(0,x) = v_0(x),\;\; v_t(0,x) = v_1(x), \end{aligned} \tag{3.1}$$

where $(t,x) \in [0,\infty) \times \mathbb{R}^n$. To study this problem we distinguish between several cases depending on the powers $r_1,\, r_2$, exponents $p,\, q$ in the power nonlinearities and the regularity parameters $s_1,\, s_2$ of the data.

If the data are low regular or taken from energy space we recall from previous chapters that the admissible range for p and q was almost defined by using the modified Fujita exponent $p_{Fuj,m}(n)$, whereas the regularity parameter of the data has no influence. For this reason in the first two sections we restrict ourselves to study only the case of data having the same regularity $s_1 = s_2 = s$. We introduce new exponents $\tilde{p}_{r_1,r_2}$ and $\tilde{q}_{r_1,r_2}$ which are generated by the interaction between the two different dissipation terms (hint of Prof. M. d'Abbicco). We treat similar cases as in Chapter 2. This means, we are interested in different cases for the modified exponents $\tilde{p} = \tilde{p}_{r_1,r_2}$ and $\tilde{q} = \tilde{q}_{r_1,r_2}$ compared with the modified Fujta exponent $p_{Fuj,m}(n)$.

On the contrary, if the data are supposed to have a high regularity we recall that the admissible range for p and q is almost defined by the regularity parameters. Then we study the effect of different regularities of data and the modified exponents are still influenced by the modified Fujita exponent. An the end of this chapter we show in which direction we can generalize our results to cover a maximal set of models with effective dissipation terms.

3.1. Low regular data

This section is devoted to treat the system (3.1), where the data are taken from the same Sobolev space $H^s(\mathbb{R}^n)$ for $s \in (0,1)$, with the same additional regularity $L^m(\mathbb{R}^n), m \in [1,2)$. For the same reason explained at the beginning of Section 2.1 of Chapter 2, we compare in our statements the introduced modified exponents $\tilde{p} = \tilde{p}_{r_1,r_2}$ and $\tilde{q} = \tilde{q}_{r_1,r_2}$ with the modified Fujita exponent $p_{Fuj,m}(n)$.

3.1.1. Both modified exponents are above the modified Fujita exponent

Theorem 3.1. *Let $n \leq \frac{4s}{2-m}$ and $n < \max\left\{\frac{2sm}{m-s}; \frac{2m(2-s)}{2-m}\right\}$. The data are supposed to belong to $\mathcal{A}_{m,s} \times \mathcal{A}_{m,s}$, $m \in [1,2)$ and $s \in (0,1)$. Moreover, the exponents p, q and the modified exponents $\tilde{p}_{r_1,r_2}$ and $\tilde{q}_{r_1,r_2}$ satisfy the following conditions:*

$$\frac{2}{m} \leq \min\{p;q\} \leq \max\{p;q\} \leq p_{GN,s}(n), \tag{3.2}$$

$$\min\{\tilde{q}_{r_1,r_2}; \tilde{p}_{r_1,r_2}\} > p_{Fuj,m}(n), \tag{3.3}$$

where

$$\tilde{q}_{r_1,r_2} = \frac{1+r_1}{1+r_2}(q-1)+1, \ \ \tilde{p}_{r_1,r_2} = \frac{1+r_2}{1+r_1}(p-1)+1.$$

Then, there exists a constant ϵ_0 such that if

$$\|(u_0,u_1)\|_{\mathcal{A}_{m,s}} + \|(v_0,v_1)\|_{\mathcal{A}_{m,s}} \leq \epsilon_0,$$

then there exists a uniquely determined global (in time) Sobolev solution to (3.1) in

$$\big(\mathcal{C}([0,\infty), H^s(\mathbb{R}^n))\big)^2.$$

Furthermore, the solution satisfies the following decay estimates:

$$\begin{aligned}
\|u(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \big(1+B_1(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)}\big(\|(u_0,u_1)\|_{\mathcal{A}_{m,s}} + \|(v_0,v_1)\|_{\mathcal{A}_{m,s}}\big),\\
\||D|^s u(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \big(1+B_1(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s}{2}}\big(\|(u_0,u_1)\|_{\mathcal{A}_{m,s}} + \|(v_0,v_1)\|_{\mathcal{A}_{m,s}}\big),\\
\|v(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \big(1+B_2(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)}\big(\|(u_0,u_1)\|_{\mathcal{A}_{m,s}} + \|(v_0,v_1)\|_{\mathcal{A}_{m,s}}\big),\\
\||D|^s v(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \big(1+B_2(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s}{2}}\big(\|(u_0,u_1)\|_{\mathcal{A}_{m,s}} + \|(v_0,v_1)\|_{\mathcal{A}_{m,s}}\big).
\end{aligned}$$

Particular case: If $n=1$, then the condition (3.2) will be changed as follows:

$$\begin{array}{ll}
\frac{2}{m} \leq \min\{p;q\} \leq \max\{p;q\} < \infty & \text{if} \quad s \in [\frac{1}{2},1),\\
\frac{2}{m} \leq \min\{p;q\} \leq \max\{p;q\} \leq p_{GN,s}(1) & \text{if} \quad s \in (0,\frac{1}{2}).
\end{array} \tag{3.4}$$

Remark 3.1. • *If $r_1 < r_2$, then $p_{Fuj,m}(n) < \tilde{q}_{r_1,r_2} < q \le p_{GN,s}(n)$. Then, to guarantee that $p_{Fuj,m}(n) < p_{GN,s}(n)$, we have to assume in the statement of Theorem 3.1 the restriction $n < \frac{2sm}{m-s}$. Conversely, if $r_2 < r_1$, then $p_{Fuj,m}(n) < \tilde{p}_{r_1,r_2} < p \le p_{GN,s}(n)$ which leads to the same restriction.*

• *It is possible to assume that one of the exponents p or q is smaller than $p_{Fuj,m}(n)$ without any loss of decay. The following example shows this effect.*

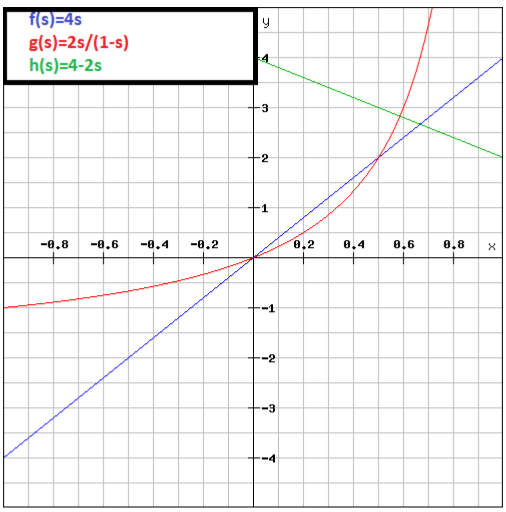
Example 3.1. *Let us assume that the dimension $n = 1$. The coefficients of the dissipation terms are $b_1(t) = (1+t)^{-\frac{1}{2}}$ and $b_2(t) = (1+t)^{\frac{1}{2}}$. The data $(u_0, u_1), (v_0, v_1)$ belong to $\mathcal{A}_{\frac{3}{2},\frac{1}{2}} \times \mathcal{A}_{\frac{3}{2},\frac{1}{2}}$. Then, the admissible range for the exponents p and q to guarantee the global (in time) existence of small data Sobolev solutions can be chosen as follows:*

$$\frac{4}{3} \le p = \frac{8}{3} < p_{Fuj,\frac{3}{2}}(1) = 4 \qquad \text{and} \qquad q = 13 \ge \frac{4}{3}.$$

Here we obtain as modified exponents $\tilde{p} = 6$ and $\tilde{q} = 5$ which satisfy the condition (3.3). The solution satisfies the following decay estimates:

$$\begin{aligned}
\|u(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim (1+t)^{-\frac{1}{24}}\big(\|(u_0,u_1)\|_{\mathcal{A}_{\frac{3}{2},\frac{1}{2}}} + \|(v_0,v_1)\|_{\mathcal{A}_{\frac{3}{2},\frac{1}{2}}}\big),\\
\||D|^s u(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim (1+t)^{-\frac{1}{6}}\big(\|(u_0,u_1)\|_{\mathcal{A}_{\frac{3}{2},\frac{1}{2}}} + \|(v_0,v_1)\|_{\mathcal{A}_{\frac{3}{2},\frac{1}{2}}}\big),\\
\|v(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim (1+t)^{-\frac{1}{8}}\big(\|(u_0,u_1)\|_{\mathcal{A}_{\frac{3}{2},\frac{1}{2}}} + \|(v_0,v_1)\|_{\mathcal{A}_{\frac{3}{2},\frac{1}{2}}}\big),\\
\||D|^s v(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim (1+t)^{-\frac{1}{2}}\big(\|(u_0,u_1)\|_{\mathcal{A}_{\frac{3}{2},\frac{1}{2}}} + \|(v_0,v_1)\|_{\mathcal{A}_{\frac{3}{2},\frac{1}{2}}}\big).
\end{aligned}$$

Remark 3.2. *The restriction of the dimension n is influenced by the regularity parameter s and the additional regularity m. This means, there is no general statement for the restrictions of the dimension n . Let us take the following example: If $m = 1$, then we have the restriction $n \le 4s$ and $n < \max\{\frac{2s}{1-s}; 4-2s\}$. The graph shows how we can get the restriction for each particular case we are interested in.*



Proof. We define the solution space $X(t)$ as follows:

$$X(t) = \left\{(u,v) \in \big(\mathcal{C}([0,t], H^s(\mathbb{R}^n))\big)^2\right\}$$

with the norm

$$\|(u,v)\|_{X(t)} = \sup_{\tau\in[0,t]} \big\{M_1(\tau,u) + M_2(\tau,v)\big\},$$

where

$$\begin{aligned} M_1(\tau,u) &= (1+B_1(\tau,0))^{\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)}\|u(\tau,\cdot)\|_{L^2(\mathbb{R}^n)} \\ &\quad +(1+B_1(\tau,0))^{\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)+\frac{s}{2}}\||D|^s u(\tau,\cdot)\|_{L^2(\mathbb{R}^n)}, \\ M_2(\tau,v) &= (1+B_2(\tau,0))^{\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)}\|v(\tau,\cdot)\|_{L^2(\mathbb{R}^n)} \\ &\quad +(1+B_2(\tau,0))^{\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)+\frac{s}{2}}\||D|v(\tau,\cdot)\|_{L^2(\mathbb{R}^n)}. \end{aligned}$$

Let N be the mapping on $X(t)$ which is defined by

$$N : (u,v) \in X(t) \to N(u,v) = (u^{ln}+u^{nl}, v^{ln}+v^{nl}), \tag{3.5}$$

where

$$\begin{aligned} u^{ln}(t,x) &:= E_{1,0}(t,0,x) *_{(x)} u_0(x) + E_{1,1}(t,0,x) *_{(x)} u_1(x), \\ u^{nl}(t,x) &:= \int_0^t E_{1,1}(t,\tau,x) *_{(x)} |v(\tau,x)|^p d\tau, \\ v^{ln}(t,x) &:= E_{2,0}(t,0,x) *_{(x)} v_0(x) + E_{2,1}(t,0,x) *_{(x)} v_1(x), \\ v^{nl}(t,x) &:= \int_0^t E_{2,1}(t,\tau,x) *_{(x)} |u(\tau,x)|^q d\tau. \end{aligned}$$

We denote by $E_{1,0} = E_{1,0}(t,0,x)$ and $E_{1,1} = E_{1,1}(t,0,x)$ the fundamental solutions to the Cauchy problem

$$u_{tt} - \Delta u + b_1(t)u_t = 0, \qquad u(0,x) = u_0(x), \qquad u_t(0,x) = u_1(x),$$

and by $E_{2,0} = E_{2,0}(t,0,x)$ and $E_{2,1} = E_{2,1}(t,0,x)$ the fundamental solutions to the the Cauchy problem

$$v_{tt} - \Delta v + b_2(t)v_t = 0, \qquad v(0,x) = v_0(x), \qquad v_t(0,x) = v_1(x).$$

Our aim is to prove the estimates

$$\|N(u,v)\|_{X(t)} \lesssim \|(u_0,u_1)\|_{\mathcal{A}_{m,s}} + \|(v_0,v_1)\|_{\mathcal{A}_{m,s}} + \|(u,v)\|_{X(t)}^p + \|(u,v)\|_{X(t)}^q, \tag{3.6}$$

$$\begin{aligned} \|N(u,v) - N(\tilde u,\tilde v)\|_{X(t)} &\lesssim \|(u,v)-(\tilde u,\tilde v)\|_{X(t)} \\ &\times\big(\|(u,v)\|_{X(t)}^{p-1} + \|(\tilde u,\tilde v)\|_{X(t)}^{p-1} + \|(u,v)\|_{X(t)}^{q-1} + \|(\tilde u,\tilde v)\|_{X(t)}^{q-1}\big). \end{aligned} \tag{3.7}$$

Let us begin to prove the first estimate (3.6). As we did in the previous chapters, we can immediately conclude the estimate for the linear part, namely

$$\|(u^{ln},v^{ln})\|_{X(t)} \lesssim \|(u_0,u_1)\|_{\mathcal{A}_{m,s}} + \|(v_0,v_1)\|_{\mathcal{A}_{m,s}}. \tag{3.8}$$

For the nonlinear part we only show how to estimate the terms $\||D|^s u^{nl}(t,\cdot)\|_{L^2(\mathbb{R}^n)}$ and $\||D|^s v^{nl}(t,\cdot)\|_{L^2(\mathbb{R}^n)}$ appearing in the definition of the norm of $X(t)$. The estimates of $\|u^{nl}\|_{L^2(\mathbb{R}^n)}$ and $\|v^{nl}\|_{L^2(\mathbb{R}^n)}$ are included there, or have no influence on the statement of the theorem. Let us begin with $\||D|^s u^{nl}\|_{L^2(\mathbb{R}^n)}$. From the estimates (1.25) of Theorem 1.5 we obtain

$$\||D|^s u^{nl}(t,\cdot)\|_{L^2(\mathbb{R}^n)} \lesssim \int_0^t b_1(\tau)^{-1}(1+B_1(t,\tau))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s}{2}}\||v(\tau,x)|^p\|_{L^m(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)}d\tau.$$

Due to the use of the Gagliardo-Nirenberg inequality we get similarly to (1.34) and (1.35) for $0 \le \tau \le t$ the following estimates:

$$\||v(\tau,x)|^p\|_{L^m(\mathbb{R}^n)} \lesssim (1+B_2(\tau,0))^{-\frac{n}{2m}p+\frac{n}{2m}}\|(u,v)\|^p_{X(t)}, \tag{3.9}$$

and

$$\||v(\tau,x)|^p\|_{L^2(\mathbb{R}^n)} \lesssim (1+B_2(\tau,0))^{-\frac{n}{2m}p+\frac{n}{4}}\|(u,v)\|^p_{X(t)}, \tag{3.10}$$

where (3.2) is supposed to be satisfied.
If $\tau \in [0,\frac{t}{2}]$, then $B(t,\tau) \approx B(t,0)$ from Lemma 1.1 and we have

$$\begin{aligned}
&\int_0^{\frac{t}{2}} b_1(\tau)^{-1}(1+B_1(t,\tau))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s}{2}}\||v(\tau,x)|^p\|_{L^m(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)}d\tau\\
&\lesssim \|(u,v)\|^p_{X(t)}\int_0^{\frac{t}{2}} b_1(\tau)^{-1}(1+B_1(t,\tau))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s}{2}}(1+B_2(\tau,0))^{-\frac{n}{2m}p+\frac{n}{2m}}d\tau\\
&\approx \|(u,v)\|^p_{X(t)}\int_0^{\frac{t}{2}} b_1(\tau)^{-1}(1+B_1(t,\tau))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s}{2}}(1+B_1(\tau,0))^{-\frac{n}{2m}(p-1)\left(\frac{1+r_2}{1+r_1}\right)}d\tau\\
&\approx \|(u,v)\|^p_{X(t)}\int_0^{\frac{t}{2}} b_1(\tau)^{-1}(1+B_1(t,\tau))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s}{2}}(1+B_1(\tau,0))^{-\frac{n}{2m}(\tilde{p}-1)}d\tau\\
&\lesssim \|(u,v)\|^p_{X(t)}(1+B_1(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s}{2}}\int_0^{\frac{t}{2}} b_1(\tau)^{-1}(1+B_1(\tau,0))^{-\frac{n}{2m}(\tilde{p}-1)}d\tau\\
&\lesssim \|(u,v)\|^p_{X(t)}(1+B_1(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s}{2}},
\end{aligned}$$

where we used $\tilde{p} > p_{Fuj,m}(n) = 1+\frac{2m}{n}$.
If $\tau \in [0,\frac{t}{2}]$, then $B(\tau,0) \approx B(t,0)$. Furthermore, we have $n < \frac{2m(2-s)}{2-m}$ from the assumptions for n. This implies $-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s}{2} > -1$. Then

$$\begin{aligned}
&\int_{\frac{t}{2}}^t b_1(\tau)^{-1}(1+B_1(t,\tau))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s}{2}}\||v(\tau,x)|^p\|_{L^m(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)}d\tau\\
&\quad\lesssim \|(u,v)\|^p_{X(t)}\int_{\frac{t}{2}}^t b_1(\tau)^{-1}(1+B_1(t,\tau))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s}{2}}(1+B_1(\tau,0))^{-\frac{n}{2m}(\tilde{p}-1)}d\tau\\
&\quad\lesssim \|(u,v)\|^p_{X(t)}(1+B_1(t,0))^{-\frac{n}{2m}(\tilde{p}-1)}\int_{\frac{t}{2}}^t b_1(\tau)^{-1}(1+B_1(t,\tau))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s}{2}}d\tau\\
&\quad\lesssim \|(u,v)\|^p_{X(t)}(1+B_1(t,0))^{-\frac{n}{2m}(\tilde{p}-1)-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s}{2}+1}\\
&\quad\lesssim \|(u,v)\|^p_{X(t)}(1+B_1(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s}{2}},
\end{aligned}$$

where we used again $\tilde{p} > p_{Fuj,m}(n)$.
Consequently, we obtain the following estimate:

$$\||D|^s u^{nl}(t,\cdot)\|_{L^2(\mathbb{R}^n)} \lesssim (1+B_1(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s}{2}}\|(u,v)\|^p_{X(t)}. \tag{3.11}$$

We get in the same way, without any additional restrictions or conditions, the following estimate:

$$\|u^{nl}(t,\cdot)\|_{L^2(\mathbb{R}^n)} \lesssim (1+B_1(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)}\|(u,v)\|^p_{X(t)}. \tag{3.12}$$

Analogously, we may obtain for $\tilde{q}_{r_1,r_2} > p_{Fuj,m}(n)$ by using (3.2) the estimates

$$\||D|^s v^{nl}(t,\cdot)\|_{L^2(\mathbb{R}^n)} \lesssim (1+B_2(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s}{2}}\|(u,v)\|^q_{X(t)}, \tag{3.13}$$

$$\|v^{nl}(t,\cdot)\|_{L^2(\mathbb{R}^n)} \lesssim (1+B_2(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)}\|(u,v)\|^q_{X(t)}. \tag{3.14}$$

From (3.11) to (3.14) we have

$$\|(u^{nl},v^{nl})\|_{X(t)} \lesssim \|(u,v)\|^p_{X(t)} + \|(u,v)\|^q_{X(t)}. \tag{3.15}$$

Finally, from (3.8) and (3.15) we get (3.6).
Now we derive the estimate (3.7). Let us assume that (u,v) and $(\tilde{u},\tilde{u})$ are two vector-functions belonging to $X(t)$. Then we have

$$\begin{aligned} N(u,v)-N(\tilde{u},\tilde{v}) = \Big(&\int_0^t E_{1,1}(t,\tau,x) *_{(x)} \left(|v(\tau,x)|^p - |\tilde{v}(\tau,x)|^p\right) d\tau, \\ &\int_0^t E_{2,1}(t,\tau,x) *_{(x)} \left(|u(\tau,x)|^q - |\tilde{u}(\tau,x)|^q\right) d\tau\Big). \end{aligned}$$

Using Hölder's inequality we obtain for $k=2,m$ the following inequalities:

$$\big\||u(\tau,\cdot)|^q - |\tilde{u}(\tau,\cdot)|^q\big\|_{L^k(\mathbb{R}^n)} \lesssim \big\|u(\tau,\cdot)-\tilde{u}(\tau,\cdot)\big\|_{L^{kq}(\mathbb{R}^n)}\big(\|u(\tau,\cdot)\|^{q-1}_{L^{kq}(\mathbb{R}^n)} + \|\tilde{u}(\tau,\cdot)\|^{q-1}_{L^{kq}(\mathbb{R}^n)}\big),$$

$$\big\||v(\tau,x)|^p - |\tilde{v}(\tau,x)|^p\big\|_{L^k(\mathbb{R}^n)} \lesssim \big\|v(\tau,\cdot)-\tilde{v}(\tau,\cdot)\big\|_{L^{kp}(\mathbb{R}^n)}\big(\|v(\tau,\cdot)\|^{p-1}_{L^{kp}(\mathbb{R}^n)} + \|\tilde{v}(\tau,\cdot)\|^{p-1}_{L^{kp}(\mathbb{R}^n)}\big).$$

Using the Gagliardo-Nirenberg inequality in form of Corollary A.1 for $k=2,m$ we obtain for $0\le\tau\le t$ the estimates:

$$\begin{aligned} \big\|v(\tau,\cdot)-\tilde{v}(\tau,\cdot)\big\|_{L^{kp}(\mathbb{R}^n)} &\lesssim (1+B_2(\tau,0))^{-\frac{n}{2m}+\frac{n}{2kp}}\|(u,v)-(\tilde{u},\tilde{v})\|_{X(t)}, \\ \|v(\tau,\cdot)\|^{p-1}_{L^{kp}(\mathbb{R}^n)} &\lesssim (1+B_2(\tau,0))^{-\frac{n}{2m}(p-1)+\frac{n}{2kp}(p-1)}\|(u,v)\|^{p-1}_{X(t)}, \\ \|\tilde{v}(\tau,\cdot)\|^{p-1}_{L^{kp}(\mathbb{R}^n)} &\lesssim (1+B_2(\tau,0))^{-\frac{n}{2m}(p-1)+\frac{n}{2kp}(p-1)}\|(\tilde{u},\tilde{v})\|^{p-1}_{X(t)}, \\ \big\|u(\tau,\cdot)-\tilde{u}(\tau,\cdot)\big\|_{L^{kq}(\mathbb{R}^n)} &\lesssim (1+B_1(\tau,0))^{-\frac{n}{2m}+\frac{n}{2kq}}\|(u,v)-(\tilde{u},\tilde{v})\|_{X(t)}, \\ \|u(\tau,\cdot)\|^{q-1}_{L^{kq}(\mathbb{R}^n)} &\lesssim (1+B_1(\tau,0))^{-\frac{n}{2m}(q-1)+\frac{n}{2kq}(q-1)}\|(u,v)\|^{q-1}_{X(t)}, \\ \|\tilde{u}(\tau,\cdot)\|^{q-1}_{L^{kq}(\mathbb{R}^n)} &\lesssim (1+B_1(\tau,0))^{-\frac{n}{2m}(q-1)+\frac{n}{2kq}(q-1)}\|(\tilde{u},\tilde{v})\|^{q-1}_{X(t)}. \end{aligned}$$

Analogously to (3.11) to (3.14) the last estimates lead to

$$\begin{aligned} &\Big\||D|^s \textstyle\int_0^t E_{1,1}(t,\tau,x) *_{(x)} \left(|v(\tau,x)|^p - |\tilde{v}(\tau,x)|^p\right) d\tau\Big\|_{L^2(\mathbb{R}^n)} \\ &\qquad \lesssim (1+B_1(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s}{2}}\|(u,v)-(\tilde{u},\tilde{v})\|_{X(t)} \\ &\qquad\quad \times\big(\|(u,v)\|^{p-1}_{X(t)} + \|(\tilde{u},\tilde{v})\|^{p-1}_{X(t)}\big), \end{aligned} \tag{3.16}$$

$$\begin{aligned} &\Big\|\textstyle\int_0^t E_{1,1}(t,\tau,x) *_{(x)} \left(|v(\tau,x)|^p - |\tilde{v}(\tau,x)|^p\right) d\tau\Big\|_{L^2(\mathbb{R}^n)} \\ &\qquad \lesssim (1+B_1(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)}\|(u,v)-(\tilde{u},\tilde{v})\|_{X(t)} \\ &\qquad\quad \times\big(\|(u,v)\|^{p-1}_{X(t)} + \|(\tilde{u},\tilde{v})\|^{p-1}_{X(t)}\big), \end{aligned} \tag{3.17}$$

$$\begin{aligned}
&\Big\| |D|^s \int_0^t E_{2,1}(t,\tau,x) *_{(x)} \big(|u(\tau,x)|^q - |\tilde{u}(\tau,x)|^q\big)\, d\tau \Big\|_{L^2(\mathbb{R}^n)} \\
&\qquad \lesssim (1+B_2(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s}{2}} \|(u,v)-(\tilde{u},\tilde{v})\|_{X(t)} \\
&\qquad\qquad \times \big(\|(u,v)\|^{q-1}_{X(t)} + \|(\tilde{u},\tilde{v})\|^{q-1}_{X(t)}\big),
\end{aligned} \tag{3.18}$$

$$\begin{aligned}
&\Big\| \int_0^t E_{2,1}(t,\tau,x) *_{(x)} \big(|u(\tau,x)|^q - |\tilde{u}(\tau,x)|^q\big)\, d\tau \Big\|_{L^2(\mathbb{R}^n)} \\
&\qquad \lesssim (1+B_2(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)} \|(u,v)-(\tilde{u},\tilde{v})\|_{X(t)} \\
&\qquad\qquad \times \big(\|(u,v)\|^{q-1}_{X(t)} + \|(\tilde{u},\tilde{v})\|^{q-1}_{X(t)}\big).
\end{aligned} \tag{3.19}$$

Using all these estimates in the norm of $X(t)$ we obtain (3.7). The proof is completed. □

3.1.2. Only one modified exponent is above the modified Fujita exponent

Theorem 3.2. *Let* $n \leq \frac{4s}{2-m}, n < \max\left\{\frac{2sm}{m-s}; \frac{2m(2-s)}{2-m}\right\}$, $r_1, r_2 \in (-1,1)$, $m \in [1,2)$ *and* $s \in (0,1)$. *The data* $(u_0,u_1),(v_0,v_1)$ *are assumed to belong to* $\mathcal{A}_{m,s} \times \mathcal{A}_{m,s}$. *Moreover, the modified exponents satisfy*

$$\tilde{p}_{r_1,r_2} < p_{Fuj,m}(n) < \tilde{q}_{r_1,r_2}, \tag{3.20}$$

$$\frac{n}{2} > m\Big(\frac{\tilde{q}_{r_1,r_2} + \frac{1+r_1}{1+r_2}}{\tilde{p}_{r_1,r_2}\tilde{q}_{r_1,r_2} - 1 + (\tilde{p}_{r_1,r_2}-1)\frac{r_1-r_2}{1+r_2}}\Big), \tag{3.21}$$

where

$$\tilde{q} := \tilde{q}_{r_1,r_2} = \frac{1+r_1}{1+r_2}(q-1)+1, \;\; \tilde{p} := \tilde{p}_{r_1,r_2} = \frac{1+r_2}{1+r_1}(p-1)+1.$$

The exponents p *and* q *of the power nonlinearities satisfy*

$$\begin{array}{llll}
\frac{2}{m} \leq \min\{p;q\} \leq \max\{p;q\} < \infty & \text{if} \;\; n=1 & \text{and} & s \in [\frac{1}{2},1), \\
\frac{2}{m} < \min\{p;q\} \leq \max\{p;q\} \leq p_{GN,s}(1) & \text{if} \;\; n=1 & \text{and} & s \in (0,\frac{1}{2}), \\
\frac{2}{m} < \min\{p;q\} \leq \max\{p;q\} \leq p_{GN,s}(n) & \text{if} & n \geq 2. &
\end{array} \tag{3.22}$$

Then, there exists a constant ϵ_0 *such that if*

$$\|(u_0,u_1)\|_{\mathcal{A}_{m,s}} + \|(v_0,v_1)\|_{\mathcal{A}_{m,s}} \leq \epsilon_0,$$

then there exists a uniquely determined global (in time) Sobolev solution to (3.1) in

$$(\mathcal{C}([0,\infty),H^s(\mathbb{R}^n)))^2.$$

Furthermore, the solution satisfies the following estimates:

$$\begin{aligned}
\|u(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \big(1+B_1(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)+\gamma_{n,m}(\tilde{p}_{r_1,r_2})} \\
&\qquad \times \big(\|(u_0,u_1)\|_{\mathcal{A}_{m,s}} + \|(v_0,v_1)\|_{\mathcal{A}_{m,s}}\big), \\
\||D|^s u(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \big(1+B_1(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s}{2}+\gamma_{n,m}(\tilde{p}_{r_1,r_2})} \\
&\qquad \times \big(\|(u_0,u_1)\|_{\mathcal{A}_{m,s}} + \|(v_0,v_1)\|_{\mathcal{A}_{m,s}}\big), \\
\|v(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \big(1+B_2(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)} \big(\|(u_0,u_1)\|_{\mathcal{A}_{m,s}} + \|(v_0,v_1)\|_{\mathcal{A}_{m,s}}\big), \\
\||D|^s v(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \big(1+B_2(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s}{2}} \big(\|(u_0,u_1)\|_{\mathcal{A}_{m,s}} + \|(v_0,v_1)\|_{\mathcal{A}_{m,s}}\big),
\end{aligned}$$

where

$$\gamma_{n,m}(\tilde{p}_{r_1,r_2}) = -\frac{n}{2m}(\tilde{p}_{r_1,r_2}-1)+1$$

represents the loss of decay in comparison with the corresponding decay estimates for the solution u to the linear Cauchy problem with vanishing right hand-side.

Remark 3.3. *If we have $p=p_{Fuj,m}(n)$ in the condition (3.20), then we obtain a small loss of decay $\gamma_{n,m}(p_{r_1,r_2})=\varepsilon$ for arbitrarily small positive ε generated by the $\log$ term which appears in the step when we control the nonlinear term of u, in particular, the integral over $[\frac{t}{2},t]$.*

Example 3.2. *Let us assume $n=2$. We choose the additional regularity $m=\frac{3}{2}$ and the regularity parameter $s=\frac{9}{10}$. Then we get $\frac{2}{m}=\frac{4}{3}$, $p_{Fuj,\frac{3}{2}}(2)=\frac{5}{2}$ and $p_{GN,\frac{9}{10}}=10$. If we take $p=\frac{19}{10}\in\left[\frac{4}{3},10\right]$ and $q=\frac{49}{9}\in\left[\frac{4}{3},10\right]$, then for $\frac{1+r_1}{1+r_2}=\frac{9}{10}$ we get*

$$\tilde{p}=2<p_{Fuj,\frac{3}{2}}(2)<5=\tilde{q}.$$

The modified exponents $\tilde{p}$ and $\tilde{q}$ satisfy condition (3.21). Moreover, the solution satisfies the following estimates:

$$\begin{aligned}
\|u(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim (1+t)^{\frac{1}{6}}\big(\|(u_0,u_1)\|_{\mathcal{A}_{\frac{3}{2},\frac{9}{10}}}+\|(v_0,v_1)\|_{\mathcal{A}_{\frac{3}{2},\frac{9}{10}}}\big),\\
\||D|^s u(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim (1+t)^{-\frac{17}{60}}\big(\|(u_0,u_1)\|_{\mathcal{A}_{\frac{3}{2},\frac{9}{10}}}+\|(v_0,v_1)\|_{\mathcal{A}_{\frac{3}{2},\frac{9}{10}}}\big),\\
\|v(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim (1+t)^{-\frac{1}{6}}\big(\|(u_0,u_1)\|_{\mathcal{A}_{\frac{3}{2},\frac{9}{10}}}+\|(v_0,v_1)\|_{\mathcal{A}_{\frac{3}{2},\frac{9}{10}}}\big),\\
\||D|^s v(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim (1+t)^{-\frac{37}{60}}\big(\|(u_0,u_1)\|_{\mathcal{A}_{\frac{3}{2},\frac{9}{10}}}+\|(v_0,v_1)\|_{\mathcal{A}_{\frac{3}{2},\frac{9}{10}}}\big).
\end{aligned}$$

Proof. We define the solution space $X(t)$ as follows:

$$X(t)=\big\{(u,v)\in\left(\mathcal{C}([0,t],H^s(\mathbb{R}^n))\right)^2\big\}$$

with the norm

$$\|(u,v)\|_{X(t)}=\sup_{\tau\in[0,t]}\big\{(1+B_1(\tau,0))^{-\gamma_{n,m}(p)}M_1(\tau,u)+M_2(\tau,v)\big\},$$

where

$$\begin{aligned}
M_1(\tau,u) &= (1+B_1(\tau,0))^{\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)}\|u(\tau,\cdot)\|_{L^2(\mathbb{R}^n)}\\
&\quad+(1+B_1(\tau,0))^{\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)+\frac{s}{2}}\||D|^s u(\tau,\cdot)\|_{L^2(\mathbb{R}^n)},\\
M_2(\tau,v) &= (1+B_2(\tau,0))^{\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)}\|v(\tau,\cdot)\|_{L^2(\mathbb{R}^n)}\\
&\quad+(1+B_2(\tau,0))^{\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)+\frac{s}{2}}\||D|v(\tau,\cdot)\|_{L^2(\mathbb{R}^n)}.
\end{aligned}$$

Let N be the mapping on $X(t)$ which is defined as in (3.5) by

$$N:(u,v)\in X(t)\to N(u,v)=(u^{ln}+u^{nl},v^{ln}+v^{nl}).$$

Our aim is to prove the following estimates:

$$\|N(u,v)\|_{X(t)}\lesssim\|(u_0,u_1)\|_{\mathcal{A}_{m,s}}+\|(v_0,v_1)\|_{\mathcal{A}_{m,s}}+\|(u,v)\|^p_{X(t)}+\|(u,v)\|^q_{X(t)},\tag{3.23}$$

$$\begin{aligned}\|N(u,v)-N(\tilde u,\tilde v)\|_{X(t)} &\lesssim \|(u,v)-(\tilde u,\tilde v)\|_{X(t)}\\ &\times\big(\|(u,v)\|^{p-1}_{X(t)}+\|(\tilde u,\tilde v)\|^{p-1}_{X(t)}+\|(u,v)\|^{q-1}_{X(t)}+\|(\tilde u,\tilde v)\|^{q-1}_{X(t)}\big).\end{aligned} \tag{3.24}$$

The estimates for the linear part of the operator N can be concluded immediately from the norm of $X(t)$ and the estimates (1.13), (1.14) of Theorem 1.2. Then we have

$$\|(u^{ln},v^{ln})\|_{X(t)} \lesssim \|(u_0,u_1)\|_{\mathcal{A}_{m,s}}+\|(v_0,v_1)\|_{\mathcal{A}_{m,s}}. \tag{3.25}$$

To estimate the nonlinear part we begin with u^{nl}. Using the estimates (3.9), (3.10) from the proof of Theorem 3.1 and the estimate (1.25) of Theorem 1.5 we get

$$\begin{aligned}
\||D|^s u^{nl}(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \int_0^t b_1(\tau)^{-1}(1+B_1(t,\tau))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s}{2}}\||v(\tau,x)|^p\|_{L^m(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)}d\tau\\
&\lesssim \|(u,v)\|^p_{X(t)}\int_0^{\frac{t}{2}} b_1(\tau)^{-1}(1+B_1(t,\tau))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s}{2}}(1+B_1(\tau,0))^{-\frac{n}{2m}(\tilde p_{r_1,r_2}-1)}d\tau\\
&\quad+\|(u,v)\|^p_{X(t)}\int_{\frac{t}{2}}^t b_1(\tau)^{-1}(1+B_1(t,\tau))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s}{2}}(1+B_1(\tau,0))^{-\frac{n}{2m}(\tilde p_{r_1,r_2}-1)}d\tau\\
&\lesssim \|(u,v)\|^p_{X(t)}(1+B_1(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s}{2}}\int_0^{\frac{t}{2}} b_1(\tau)^{-1}(1+B_1(\tau,0))^{-\frac{n}{2m}(\tilde p_{r_1,r_2}-1)}d\tau\\
&\quad+\|(u,v)\|^p_{X(t)}(1+B_1(t,0))^{-\frac{n}{2m}(\tilde p_{r_1,r_2}-1)}\int_{\frac{t}{2}}^t b_1(\tau)^{-1}(1+B_1(t,\tau))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s}{2}}d\tau.
\end{aligned}$$

If $\tau\in[0,\frac{t}{2}]$, then

$$\int_0^{\frac{t}{2}} b_1(\tau)^{-1}(1+B_1(\tau,0))^{-\frac{n}{2m}(\tilde p_{r_1,r_2}-1)}d\tau \lesssim (1+B_1(t,0))^{\gamma_{n,m}(\tilde p_{r_1,r_2})}.$$

If $\tau\in[\frac{t}{2},t]$ and $-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s}{2}>-1$, then

$$\int_{\frac{t}{2}}^t b_1(\tau)^{-1}(1+B_1(t,\tau))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s}{2}}d\tau \lesssim (1+B_1(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s}{2}+1}.$$

Consequently, we get

$$\||D|^s u^{nl}(t,\cdot)\|_{L^2(\mathbb{R}^n)} \lesssim \|(u,v)\|^p_{X(t)}(1+B_1(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s}{2}+\gamma_{n,m}(\tilde p_{r_1,r_2})}. \tag{3.26}$$

In the same way we obtain

$$\|u^{nl}(t,\cdot)\|_{L^2(\mathbb{R}^n)} \lesssim \|(u,v)\|^p_{X(t)}(1+B_1(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)+\gamma_{n,m}(\tilde p_{r_1,r_2})}. \tag{3.27}$$

On the other hand, for v^{nl} we have

$$\begin{aligned}
&\||D|^s v^{nl}(t,\cdot)\|_{L^2(\mathbb{R}^n)} \\
&\lesssim \int_0^t b_2(\tau)^{-1}(1+B_2(t,\tau))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s}{2}}\||u(\tau,\cdot)|^q\|_{L^m(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)}d\tau \\
&\lesssim \|(u,v)\|^q_{X(t)}\int_0^{\frac{t}{2}} b_2(\tau)^{-1}(1+B_2(t,\tau))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s}{2}} \\
&\qquad\qquad \times(1+B_2(\tau,0))^{-\frac{n}{2m}(\tilde{q}_{r_1,r_2}-1)+\gamma_{n,m}(\tilde{p}_{r_1,r_2})q\left(\frac{1+r_1}{1+r_2}\right)}d\tau \\
&\quad +\|(u,v)\|^q_{X(t)}\int_{\frac{t}{2}}^t b_2(\tau)^{-1}(1+B_2(t,\tau))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s}{2}} \\
&\qquad\qquad \times(1+B_2(\tau,0))^{-\frac{n}{2m}(\tilde{q}_{r_1,r_2}-1)+\gamma_{n,m}(\tilde{p}_{r_1,r_2})q\left(\frac{1+r_1}{1+r_2}\right)}d\tau.
\end{aligned}$$

If $\tau\in[0,\frac{t}{2}]$, then from (1.6) we have

$$\begin{aligned}
&\int_0^{\frac{t}{2}} b_2(\tau)^{-1}(1+B_2(t,\tau))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s}{2}}(1+B_2(\tau,0))^{-\frac{n}{2m}(\tilde{q}_{r_1,r_2}-1)+\gamma_{n,m}(\tilde{p}_{r_1,r_2})q\left(\frac{1+r_1}{1+r_2}\right)}d\tau \\
&\quad\lesssim (1+B_2(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s}{2}},
\end{aligned}$$

where we use

$$-\frac{n}{2m}(\tilde{q}_{r_1,r_2}-1)+\gamma_{n,m}(\tilde{p}_{r_1,r_2})q\Big(\frac{1+r_1}{1+r_2}\Big)+1<0$$

which is equivalent to (3.21). Indeed, we have

$$\begin{aligned}
&-\frac{n}{2m}(\tilde{q}_{r_1,r_2}-1)+\gamma_{n,m}(\tilde{p}_{r_1,r_2})q\Big(\frac{1+r_1}{1+r_2}\Big)+1<0 \\
&\quad\Leftrightarrow -\frac{n}{2m}(\tilde{q}_{r_1,r_2}-1)+\gamma_{n,m}(\tilde{p}_{r_1,r_2})\left(\frac{1+r_2}{1+r_1}(\tilde{q}_{r_1,r_2}-1)+1\right)\left(\frac{1+r_1}{1+r_2}\right)+1<0 \\
&\quad\Leftrightarrow -\frac{n}{2m}(\tilde{q}_{r_1,r_2}-1)+\gamma_{n,m}(\tilde{p}_{r_1,r_2})(\tilde{q}_{r_1,r_2}-1)+\gamma_{n,m}(\tilde{p}_{r_1,r_2})\Big(\frac{1+r_1}{1+r_2}\Big)+1<0 \\
&\quad\Leftrightarrow -\frac{n}{2m}\left(\tilde{q}_{r_1,r_2}-1+(\tilde{p}_{r_1,r_2}-1)(\tilde{q}_{r_1,r_2}-1)+(\tilde{p}_{r_1,r_2}-1)\Big(\frac{1+r_1}{1+r_2}\Big)\right) \\
&\qquad\qquad\qquad\qquad +\tilde{q}_{r_1,r_2}+\frac{1+r_1}{1+r_2}<0 \\
&\quad\Leftrightarrow -\frac{n}{2m}\left(\tilde{p}_{r_1,r_2}\tilde{q}_{r_1,r_2}-1+(\tilde{p}_{r_1,r_2}-1)\frac{r_1-r_2}{1+r_2}\right)+\tilde{q}_{r_1,r_2}+\frac{1+r_1}{1+r_2}<0.
\end{aligned}$$

Then, we obtain

$$\begin{aligned}
&-\tfrac{n}{2m}(\tilde{q}_{r_1,r_2}-1)+\gamma_{n,m}(\tilde{p}_{r_1,r_2})q\Big(\tfrac{1+r_1}{1+r_2}\Big)+1<0 \\
&\qquad\Updownarrow \\
&\tfrac{n}{2}>m\Big(\tfrac{\tilde{q}_{r_1,r_2}+\frac{1+r_1}{1+r_2}}{\tilde{p}_{r_1,r_2}\tilde{q}_{r_1,r_2}-1+(\tilde{p}_{r_1,r_2}-1)\frac{r_1-r_2}{1+r_2}}\Big).
\end{aligned}\tag{3.28}$$

If $\tau \in [\frac{t}{2}, t]$, then from (1.7) we have

$$\begin{aligned}
&\int_{\frac{t}{2}}^{t} b_2(\tau)^{-1}(1+B_2(t,\tau))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s}{2}}(1+B_2(\tau,0))^{-\frac{n}{2m}(\tilde{q}_{r_1,r_2}-1)+\gamma_{n,m}(\tilde{p}_{r_1,r_2})q\left(\frac{1+r_1}{1+r_2}\right)}d\tau \\
&\lesssim (1+B_2(t,0))^{-\frac{n}{2m}(\tilde{q}_{r_1,r_2}-1)+\gamma_{n,m}(\tilde{p}_{r_1,r_2})q\left(\frac{1+r_1}{1+r_2}\right)}\int_{\frac{t}{2}}^{t} b_2(\tau)^{-1}(1+B_2(t,\tau))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s}{2}}d\tau \\
&\lesssim (1+B_2(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s}{2}},
\end{aligned}$$

where we used again

$$-\frac{n}{2m}(\tilde{q}_{r_1,r_2}-1)+\gamma_{n,m}(\tilde{p})q\Big(\frac{1+r_1}{1+r_2}\Big)+1<0.$$

Consequently, we have

$$\||D|^s v^{nl}(t,\cdot)\|_{L^2(\mathbb{R}^n)} \lesssim \|(u,v)\|_{X(t)}^q (1+B_2(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s}{2}}. \tag{3.29}$$

In the same way we may conclude

$$\|v^{nl}(t,\cdot)\|_{L^2(\mathbb{R}^n)} \lesssim \|(u,v)\|_{X(t)}^q (1+B_2(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)}. \tag{3.30}$$

Finally, from (3.26) to (3.30) we get

$$\|(u^{nl},v^{nl})\|_{X(t)} \lesssim \|(u,v)\|_{X(t)}^p + \|(u,v)\|_{X(t)}^q. \tag{3.31}$$

We conclude the estimate (3.23) from (3.25) and (3.31).
Now we prove (3.24). Let us assume that (u,v) and $(\tilde{u},\tilde{u})$ are two vector-functions belonging to $X(t)$. Then we have

$$\begin{aligned}
N(u,v)-N(\tilde{u},\tilde{v}) = \Big(&\int_0^t E_{1,1}(t,\tau,x) *_{(x)} \big(|v(\tau,x)|^p - |\tilde{v}(\tau,x)|^p\big)\,d\tau, \\
&\int_0^t E_{2,1}(t,\tau,x) *_{(x)} \big(|u(\tau,x)|^q - |\tilde{u}(\tau,x)|^q\big)\,d\tau\Big).
\end{aligned}$$

Using Hölder's inequality we obtain for $k=2,m$ the following inequalities:

$$\big\||u(\tau,\cdot)|^q-|\tilde{u}(\tau,\cdot)|^q\big\|_{L^k(\mathbb{R}^n)} \lesssim \big\|u(\tau,\cdot)-\tilde{u}(\tau,\cdot)\big\|_{L^{kq}(\mathbb{R}^n)}\big(\|u(\tau,\cdot)\|_{L^{kq}(\mathbb{R}^n)}^{q-1}+\|\tilde{u}(\tau,\cdot)\|_{L^{kq}(\mathbb{R}^n)}^{q-1}\big),$$

$$\big\||v(\tau,x)|^p-|\tilde{v}(\tau,x)|^p\big\|_{L^k(\mathbb{R}^n)} \lesssim \big\|v(\tau,\cdot)-\tilde{v}(\tau,\cdot)\big\|_{L^{kp}(\mathbb{R}^n)}\big(\|v(\tau,\cdot)\|_{L^{kp}(\mathbb{R}^n)}^{p-1}+\|\tilde{v}(\tau,\cdot)\|_{L^{kp}(\mathbb{R}^n)}^{p-1}\big).$$

Using the Gagliardo-Nirenberg inequality for $k=2,m$ and $0\le\tau\le t$ we obtain

$$\begin{aligned}
\big\|u(\tau,\cdot)-\tilde{u}(\tau,\cdot)\big\|_{L^{kq}(\mathbb{R}^n)} &\lesssim (1+B_1(\tau,0))^{-\frac{n}{2m}+\frac{n}{2kq}+\gamma_{n,m}(\tilde{p}_{r_1,r_2})q}\sup_{\tau\in[0,t]} M_1(\tau,u-\tilde{u}), \\
\|u(\tau,\cdot)\|_{L^{kq}(\mathbb{R}^n)}^{q-1} &\lesssim (1+B_1(\tau,0))^{\left(-\frac{n}{2m}+\frac{n}{2kq}+\gamma_{n,m}(\tilde{p}_{r_1,r_2})\right)(q-1)}\|(u,v)\|_{X(t)}^{q-1}, \\
\|\tilde{u}(\tau,\cdot)\|_{L^{kq}(\mathbb{R}^n)}^{q-1} &\lesssim (1+B_1(\tau,0))^{\left(-\frac{n}{2m}+\frac{n}{2kq}+\gamma_{n,m}(\tilde{p}_{r_1,r_2})\right)(q-1)}\|(\tilde{u},\tilde{v})\|_{X(t)}^{q-1}, \\
\big\|v(\tau,\cdot)-\tilde{v}(\tau,\cdot)\big\|_{L^{kp}(\mathbb{R}^n)} &\lesssim (1+B_2(\tau,0))^{-\frac{n}{2m}+\frac{n}{2kp}}\sup_{\tau\in[0,t]} M_2(\tau,v-\tilde{v}), \\
\|v(\tau,\cdot)\|_{L^{kp}(\mathbb{R}^n)}^{p-1} &\lesssim (1+B_2(\tau,0))^{-\frac{n}{2m}(p-1)+\frac{n}{2kp}(p-1)}\|(u,v)\|_{X(t)}^{p-1}, \\
\|\tilde{v}(\tau,\cdot)\|_{L^{kp}(\mathbb{R}^n)}^{p-1} &\lesssim (1+B_2(\tau,0))^{-\frac{n}{2m}(p-1)+\frac{n}{2kp}(p-1)}\|(\tilde{u},\tilde{v})\|_{X(t)}^{p-1}.
\end{aligned}$$

Analogously to (3.26), (3.27), (3.29) and (3.30) by using the last estimates we may conclude

$$\begin{aligned}\Big\||D|^s \textstyle\int_0^t E_{1,1}(t,\tau,x) *_{(x)} \big(|v(\tau,x)|^p - |\tilde{v}(\tau,x)|^p\big)\, d\tau\Big\|_{L^2(\mathbb{R}^n)} \\ \lesssim (1+B_1(t,0))^{-\frac{n}{2}(\frac{1}{m}-\frac{1}{2})-\frac{s}{2}+\gamma_{n,m}(\tilde{p}_{r_1,r_2})} \\ \times\|(u,v)-(\tilde{u},\tilde{v})\|_{X(t)}\big(\|(u,v)\|_{X(t)}^{p-1}+\|(\tilde{u},\tilde{v})\|_{X(t)}^{p-1}\big),\end{aligned} \tag{3.32}$$

$$\begin{aligned}\Big\| \textstyle\int_0^t E_{1,1}(t,\tau,x) *_{(x)} \big(|v(\tau,x)|^p - |\tilde{v}(\tau,x)|^p\big)\, d\tau\Big\|_{L^2(\mathbb{R}^n)} \\ \lesssim (1+B_1(t,0))^{-\frac{n}{2}(\frac{1}{m}-\frac{1}{2})+\gamma_{n,m}(\tilde{p}_{r_1,r_2})} \\ \times\|(u,v)-(\tilde{u},\tilde{v})\|_{X(t)}\big(\|(u,v)\|_{X(t)}^{p-1}+\|(\tilde{u},\tilde{v})\|_{X(t)}^{p-1}\big),\end{aligned} \tag{3.33}$$

$$\begin{aligned}\Big\||D|^s \textstyle\int_0^t E_{2,1}(t,\tau,x) *_{(x)} \big(|u(\tau,x)|^q - |\tilde{u}(\tau,x)|^q\big)\, d\tau\Big\|_{L^2(\mathbb{R}^n)} \\ \lesssim (1+B_2(t,0))^{-\frac{n}{2}(\frac{1}{m}-\frac{1}{2})-\frac{s}{2}} \\ \times\|(u,v)-(\tilde{u},\tilde{v})\|_{X(t)}\big(\|(u,v)\|_{X(t)}^{q-1}+\|(\tilde{u},\tilde{v})\|_{X(t)}^{q-1}\big),\end{aligned} \tag{3.34}$$

$$\begin{aligned}\Big\| \textstyle\int_0^t E_{2,1}(t,\tau,x) *_{(x)} \big(|u(\tau,x)|^q - |\tilde{u}(\tau,x)|^q\big)\, d\tau\Big\|_{L^2(\mathbb{R}^n)} \\ \lesssim (1+B_2(t,0))^{-\frac{n}{2}(\frac{1}{m}-\frac{1}{2})} \\ \times\|(u,v)-(\tilde{u},\tilde{v})\|_{X(t)}\big(\|(u,v)\|_{X(t)}^{q-1}+\|(\tilde{u},\tilde{v})\|_{X(t)}^{q-1}\big).\end{aligned} \tag{3.35}$$

In this way we complete the proof. □

If we consider the opposite case to the case which is discussed in Theorem 3.2, that is, if $\tilde{q}_{r_1,r_2} < p_{Fuj,m}(n) < \tilde{p}_{r_1,r_2}$, then we get the following corresponding result to Theorem 3.2.

Theorem 3.3. *Let $n \leq \frac{4s}{2-m}, n < \max\big\{\frac{2sm}{m-s}; \frac{2m(2-s)}{2-m}\big\}$, $r_1, r_2 \in (-1,1)$, $m \in [1,2)$, and $s \in (0,1)$. The data $(u_0,u_1),(v_0,v_1)$ are assumed to belong to $\mathcal{A}_{m,s}\times\mathcal{A}_{m,s}$. Moreover, let the modified exponents satisfy*

$$\tilde{q}_{r_1,r_2} < p_{Fuj,m}(n) < \tilde{p}_{r_1,r_2},$$

$$\frac{n}{2} > m\Big(\frac{\tilde{p}_{r_1,r_2}+\frac{1+r_2}{1+r_1}}{\tilde{p}_{r_1,r_2}\tilde{q}_{r_1,r_2}-1+(\tilde{q}_{r_1,r_2}-1)\frac{r_2-r_1}{1+r_1}}\Big), \tag{3.36}$$

where

$$\tilde{q}_{r_1,r_2} = \frac{1+r_1}{1+r_2}(q-1)+1,\ \ \tilde{p}_{r_1,r_2} = \frac{1+r_2}{1+r_1}(p-1)+1.$$

The exponents p and q of the power nonlinearities satisfy

$$\begin{array}{llll} \frac{2}{m} \leq \min\{p;q\} \leq \max\{p;q\} < \infty & \text{if } \ n=1 & \text{and} & s\in[\frac{1}{2},1), \\ \frac{2}{m} \leq \min\{p;q\} \leq \max\{p;q\} \leq p_{GN,s}(1) & \text{if } \ n=1 & \text{and} & s\in(0,\frac{1}{2}), \\ \frac{2}{m} \leq \min\{p;q\} \leq \max\{p;q\} \leq p_{GN,s}(n) & \text{if} & & n\geq 2. \end{array}$$

Then, there exists a constant ϵ_0 such that if

$$\|(u_0,u_1)\|_{\mathcal{A}_{m,s}} + \|(v_0,v_1)\|_{\mathcal{A}_{m,s}} \leq \epsilon_0,$$

then there exists a uniquely determined global (in time) Sobolev solution to (3.1) in

$$(\mathcal{C}([0,\infty),H^s(\mathbb{R}^n)))^2.$$

Furthermore, the solution satisfies the following estimates:

$$\|u(t,\cdot)\|_{L^2(\mathbb{R}^n)} \lesssim \big(1+B_1(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)}\big(\|(u_0,u_1)\|_{\mathcal{A}_{m,s}}+\|(v_0,v_1)\|_{\mathcal{A}_{m,s}}\big),$$

$$\||D|^s u(t,\cdot)\|_{L^2(\mathbb{R}^n)} \lesssim \big(1+B_1(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s}{2}}\big(\|(u_0,u_1)\|_{\mathcal{A}_{m,s}}+\|(v_0,v_1)\|_{\mathcal{A}_{m,s}}\big),$$

$$\|v(t,\cdot)\|_{L^2(\mathbb{R}^n)} \lesssim \big(1+B_2(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)+\gamma_{n,m}(\tilde{q}_{r_1,r_2})}\big(\|(u_0,u_1)\|_{\mathcal{A}_{m,s}}+\|(v_0,v_1)\|_{\mathcal{A}_{m,s}}\big),$$

$$\||D|^s v(t,\cdot)\|_{L^2(\mathbb{R}^n)} \lesssim \big(1+B_2(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s}{2}+\gamma_{n,m}(\tilde{p}_{r_1,r_2})}\big(\|(u_0,u_1)\|_{\mathcal{A}_{m,s}}+\|(v_0,v_1)\|_{\mathcal{A}_{m,s}}\big),$$

where

$$\gamma_{n,m}(\tilde{q}_{r_1,r_2}) = -\frac{n}{2m}(\tilde{q}_{r_1,r_2}-1)+1$$

represents the loss of decay in comparison with the corresponding decay estimates for the solution v to the linear Cauchy problem with vanishing right hand-side.

Comparing with Theorem 3.2, we remark that the loss of decay in Theorem 3.3 appears with respect to v. Taking this loss of decay into consideration we have to satisfy the following condition:

$$-\frac{n}{2m}(\tilde{p}_{r_1,r_2}-1)+\gamma_{n,m}(\tilde{q}_{r_1,r_2})p\Big(\frac{1+r_2}{1+r_1}\Big)+1<0$$

which is equivalent to (3.36).

Summarizing the statements of Theorems 3.2 and 3.3 we conclude the following corollary.

Corollary 3.1. *Let $n \leq \frac{4s}{2-m}, n < \max\big\{\frac{2sm}{m-s};\frac{2m(2-s)}{2-m}\big\}$, $r_1,r_2\in(-1,1)$, $m\in[1,2)$ and $s\in(0,1)$. The data $(u_0,u_1),(v_0,v_1)$ are assumed to belong to $\mathcal{A}_{m,s}\times\mathcal{A}_{m,s}$. Moreover, let the modified exponents satisfy*

$$\min\{\tilde{p}_{r_1,r_2};\tilde{q}_{r_1,r_2}\} < p_{Fuj,m}(n) < \max\{\tilde{p}_{r_1,r_2};\tilde{q}_{r_1,r_2}\},$$

$$\frac{n}{2} > m\Big(\frac{\max\{\tilde{p}_{r_1,r_2};\tilde{q}_{r_1,r_2}\}+\gamma}{\min\{\tilde{p}_{r_1,r_2};\tilde{q}_{r_1,r_2}\}\times\max\{\tilde{p}_{r_1,r_2};\tilde{q}_{r_1,r_2}\}-1+(\min\{\tilde{p}_{r_1,r_2};\tilde{q}_{r_1,r_2}\}-1)\delta}\Big),$$

where

$$\tilde{q}_{r_1,r_2} = \frac{1+r_1}{1+r_2}(q-1)+1,\ \ \tilde{p}_{r_1,r_2} = \frac{1+r_2}{1+r_1}(p-1)+1$$

and

$$\gamma = \frac{1+r_1}{1+r_2},\ \ \delta = \frac{r_1-r_2}{1+r_2}\quad \textit{if}\ \ \tilde{p}_{r_1,r_2} < \tilde{q}_{r_1,r_2},$$
$$\gamma = \frac{1+r_2}{1+r_1},\ \ \delta = \frac{r_2-r_1}{1+r_1}\quad \textit{if}\ \ \tilde{q}_{r_1,r_2} < \tilde{p}_{r_1,r_1}.$$

The exponents p and q of the power nonlinearities satisfy

$$\begin{array}{llll} \frac{2}{m} \leq \min\{p;q\} \leq \max\{p;q\} < \infty & \text{if} \quad n=1 & \text{and} & s \in [\frac{1}{2},1), \\ \frac{2}{m} < \min\{p;q\} \leq \max\{p;q\} \leq p_{GN,s}(1) & \text{if} \quad n=1 & \text{and} & s \in (0,\frac{1}{2}), \\ \frac{2}{m} < \min\{p;q\} \leq \max\{p;q\} \leq p_{GN,s}(n) & \text{if} & n \geq 2. & \end{array}$$

Then, there exists a constant ϵ_0 such that if

$$\|(u_0,u_1)\|_{\mathcal{A}_{m,s}} + \|(v_0,v_1)\|_{\mathcal{A}_{m,s}} \leq \epsilon_0,$$

then there exists a uniquely determined global (in time) Sobolev solution to (3.1) in

$$(\mathcal{C}([0,\infty),H^s(\mathbb{R}^n)))^2.$$

Furthermore, the solution satisfies the following estimates:

$$\|u(t,\cdot)\|_{L^2(\mathbb{R}^n)} \lesssim \big(1+B_1(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)+[\gamma_{n,m}(\tilde{p}_{r_1,r_2})]^+}\big(\|(u_0,u_1)\|_{\mathcal{A}_{m,s}}+\|(v_0,v_1)\|_{\mathcal{A}_{m,s}}\big),$$

$$\||D|^s u(t,\cdot)\|_{L^2(\mathbb{R}^n)} \lesssim \big(1+B_1(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s}{2}+[\gamma_{n,m}(\tilde{p}_{r_1,r_2})]^+}\big(\|(u_0,u_1)\|_{\mathcal{A}_{m,s}}+\|(v_0,v_1)\|_{\mathcal{A}_{m,s}}\big),$$

$$\|v(t,\cdot)\|_{L^2(\mathbb{R}^n)} \lesssim \big(1+B_2(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)+[\gamma_{n,m}(\tilde{q}_{r_1,r_2})]^+}\big(\|(u_0,u_1)\|_{\mathcal{A}_{m,s}}+\|(v_0,v_1)\|_{\mathcal{A}_{m,s}}\big),$$

$$\||D|^s v(t,\cdot)\|_{L^2(\mathbb{R}^n)} \lesssim \big(1+B_2(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s}{2}+[\gamma_{n,m}(\tilde{q}_{r_1,r_2})]^+}\big(\|(u_0,u_1)\|_{\mathcal{A}_{m,s}}+\|(v_0,v_1)\|_{\mathcal{A}_{m,s}}\big),$$

where

$$[\gamma_{n,m}(\tilde{p}_{r_1,r_2})]^+ = \max\Big\{-\frac{n}{2m}(\tilde{p}_{r_1,r_2}-1)+1;0\Big\},$$

$\Big($ *resp.*

$$[\gamma_{n,m}(\tilde{q}_{r_1,r_2})]^+ = \max\Big\{-\frac{n}{2m}(\tilde{q}_{r_1,r_2}-1)+1;0\Big\}\Big)$$

represents the loss of decay in comparison with the corresponding decay estimates for the solution u (resp. v) to the linear Cauchy problem with vanishing right hand-side.

3.2. Data from the energy space

Similarly to the previous section, we treat in this section the limit case of Section 3.1, that is, the data $(u_0,u_1),(v_0,v_1)$ are supposed to belong to $\mathcal{A}_{m,1}\times\mathcal{A}_{m,1}$. Therefore, we can prove a global (in time) existence results of small data energy solutions. Now we observe that $-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-1$ cannot be larger than -1 in any condition. This forces us to include the additional regularity parameter m in the definition of the modified exponents $\tilde{p}$ or $\tilde{q}$ of the power nonlinearities.

3.2.1. Both modified exponents are above the modified Fujita exponent

Theorem 3.4. *Let $n \leq \frac{4}{2-m}$, $n < \frac{2m}{m-1}$, $-1 < r_1 < r_2 < 1$ and $m \in [1,2)$. The data $(u_0,u_1),(v_0,v_1)$ are assumed to belong to $\mathcal{A}_{m,1} \times \mathcal{A}_{m,1}$. Moreover, let the modified exponents satisfy*

$$\min\{\tilde{q}_{r_1,r_2,m};\tilde{p}_{r_1,r_2}\} > p_{Fuj,m}(n), \tag{3.37}$$

where

$$\tilde{q} := \tilde{q}_{r_1,r_2,m} = \frac{1+r_1}{1+r_2}\Big(q-\frac{m}{2}\Big)+\frac{m}{2},\ \ \tilde{p} := \tilde{p}_{r_1,r_2} = \frac{1+r_2}{1+r_1}(p-1)+1.$$

The exponents p and q of the power nonlinearities satisfy

$$\begin{array}{ll} \frac{2}{m} \leq \min\{p;q\} \leq \max\{p;q\} < \infty & \text{if } \ n \leq 2, \\ \frac{2}{m} \leq \min\{p;q\} \leq \max\{p;q\} \leq p_{GN}(n) & \text{if } \ n > 2. \end{array} \tag{3.38}$$

Then, there exists a constant ϵ_0 such that if

$$\|(u_0,u_1)\|_{\mathcal{A}_{m,1}} + \|(v_0,v_1)\|_{\mathcal{A}_{m,1}} \leq \epsilon_0,$$

then there exists a uniquely determined global (in time) energy solution to (3.1) in

$$\Big(\mathcal{C}\big([0,\infty),H^1(\mathbb{R}^n)\big)\cap\mathcal{C}^1\big([0,\infty),L^2(\mathbb{R}^n)\big)\Big)^2.$$

Furthermore, the solution satisfies the following decay estimates:

$$\|u(t,\cdot)\|_{L^2(\mathbb{R}^n)} \lesssim \big(1+B_1(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)}\big(\|(u_0,u_1)\|_{\mathcal{A}_{m,1}}+\|(v_0,v_1)\|_{\mathcal{A}_{m,1}}\big),$$

$$\|\nabla u(t,\cdot)\|_{L^2(\mathbb{R}^n)} \lesssim \big(1+B_1(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{1}{2}}\big(\|(u_0,u_1)\|_{\mathcal{A}_{m,1}}+\|(v_0,v_1)\|_{\mathcal{A}_{m,1}}\big),$$

$$\|u_t(t,\cdot)\|_{L^2(\mathbb{R}^n)} \lesssim b_1(t)^{-1}\big(1+B_1(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-1}\big(\|(u_0,u_1)\|_{\mathcal{A}_{m,1}}+\|(v_0,v_1)\|_{\mathcal{A}_{m,1}}\big),$$

$$\|v(t,\cdot)\|_{L^2(\mathbb{R}^n)} \lesssim \big(1+B_2(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)}\big(\|(u_0,u_1)\|_{\mathcal{A}_{m,1}}+\|(v_0,v_1)\|_{\mathcal{A}_{m,1}}\big),$$

$$\|\nabla v(t,\cdot)\|_{L^2(\mathbb{R}^n)} \lesssim \big(1+B_2(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{1}{2}}\big(\|(u_0,u_1)\|_{\mathcal{A}_{m,1}}+\|(v_0,v_1)\|_{\mathcal{A}_{m,1}}\big),$$

$$\|v_t(t,\cdot)\|_{L^2(\mathbb{R}^n)} \lesssim b_2(t)^{-1}\big(1+B_2(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-1}\big(\|(u_0,u_1)\|_{\mathcal{A}_{m,1}}+\|(v_0,v_1)\|_{\mathcal{A}_{m,1}}\big).$$

Example 3.3. *Let us choose the dimension $n=2$. The coefficients of the dissipation terms are $b_1(t)=(1+t)^{-\frac{1}{2}}$ and $b_2(t)=(1+t)^{\frac{1}{2}}$. The data $(u_0,u_1),(v_0,v_1)$ are supposed to belong to $\mathcal{A}_{\frac{3}{2},1}\times\mathcal{A}_{\frac{3}{2},1}$. Then the admissible range for the exponents p and q to get a global (in time) existence result of small data energy solutions can be chosen as follows:*

$$\frac{4}{3} \leq p = 2 < p_{Fuj,\frac{3}{2}}(2) = \frac{5}{2} \qquad \text{and} \qquad q = 7 \geq \frac{4}{3}.$$

From the definition of the modified exponents we obtain $\tilde{p}=4$ and $\tilde{q}=\frac{11}{4}$. Hence, condition (3.37) is satisfied. The solution satisfies the following decay estimates:

$$\|u(t,\cdot)\|_{L^2(\mathbb{R}^n)} \lesssim (1+t)^{-\frac{1}{12}}\big(\|(u_0,u_1)\|_{\mathcal{A}_{m,1}}+\|(v_0,v_1)\|_{\mathcal{A}_{m,1}}\big),$$

$$\begin{aligned}
\|\nabla u(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim (1+t)^{-\frac{1}{3}}\big(\|(u_0,u_1)\|_{\mathcal{A}_{m,1}}+\|(v_0,v_1)\|_{\mathcal{A}_{m,1}}\big),\\
\|u_t(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim (1+t)^{-\frac{1}{12}}\big(\|(u_0,u_1)\|_{\mathcal{A}_{m,1}}+\|(v_0,v_1)\|_{\mathcal{A}_{m,1}}\big),\\
\|v(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim (1+t)^{-\frac{1}{4}}\big(\|(u_0,u_1)\|_{\mathcal{A}_{m,1}}+\|(v_0,v_1)\|_{\mathcal{A}_{m,1}}\big),\\
\|\nabla v(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim (1+t)^{-1}\big(\|(u_0,u_1)\|_{\mathcal{A}_{m,1}}+\|(v_0,v_1)\|_{\mathcal{A}_{m,1}}\big),\\
\|v_t(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim (1+t)^{-\frac{9}{4}}\big(\|(u_0,u_1)\|_{\mathcal{A}_{m,1}}+\|(v_0,v_1)\|_{\mathcal{A}_{m,1}}\big).
\end{aligned}$$

Proof. We can prove this theorem by following the same steps used in the proof of Theorem 3.1, after setting $s=1$. But in this case we define a modified solution space $X(t)$ with a modified norm containing additional terms formed by suitable norms of u_t and v_t. Namely, we choose

$$X(t)=\big\{(u,v)\in\big(\mathcal{C}([0,t],H^1(\mathbb{R}^n))\cap\mathcal{C}^1([0,t],L^2(\mathbb{R}^n))\big)^2\big\}$$

with the norm

$$\|(u,v)\|_{X(t)}=\sup_{\tau\in[0,t]}\big\{M_1(\tau,u)+M_2(\tau,v)\big\},$$

where

$$\begin{aligned}
M_1(\tau,u) &= (1+B_1(\tau,0))^{\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)}\|u(\tau,\cdot)\|_{L^2(\mathbb{R}^n)}\\
&\quad+(1+B_1(\tau,0))^{\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)+\frac{s}{2}}\||D|^s u(\tau,\cdot)\|_{L^2(\mathbb{R}^n)}\\
&\quad+b_1(\tau)(1+B_1(\tau,0))^{\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)+1}\|u_t(\tau,\cdot)\|_{L^2(\mathbb{R}^n)},\\
M_2(\tau,v) &= (1+B_2(\tau,0))^{\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)}\|v(\tau,\cdot)\|_{L^2(\mathbb{R}^n)}\\
&\quad+(1+B_2(\tau,0))^{\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)+\frac{s}{2}}\||D|^s v(\tau,\cdot)\|_{L^2(\mathbb{R}^n)}\\
&\quad+b_2(\tau)(1+B_2(\tau,0))^{\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)+1}\|v_t(\tau,\cdot)\|_{L^2(\mathbb{R}^n)}.
\end{aligned}$$

Additionally to the norms, which are controlled in Theorem 3.1, for $s=1$ we control the norms $\big\|u_t^{nl}(t,\cdot)\big\|_{L^2(\mathbb{R}^n)}$ and $\big\|v_t^{nl}(t,\cdot)\big\|_{L^2(\mathbb{R}^n)}$, too. Indeed, from the estimate (1.23) in Theorem 1.4 we get

$$\big\|u_t^{nl}(t,\cdot)\big\|_{L^2(\mathbb{R}^n)}\lesssim\int_0^t b_1(\tau)^{-1}b_1(t)^{-1}(1+B_1(t,\tau))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-1}\||v(\tau,x)|^p\|_{L^m(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)}d\tau.$$

Using (3.9), (3.10) and the relation (1.6) for the integral over $[0,\frac{t}{2}]$ we have

$$\begin{aligned}
&\|(u,v)\|^p_{X(t)}\int_0^{\frac{t}{2}} b_1(\tau)^{-1}b_1(t)^{-1}(1+B_1(t,\tau))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-1}(1+B_2(\tau,0))^{-\frac{n}{2m}(p-1)}d\tau\\
&\lesssim \|(u,v)\|^p_{X(t)}\int_0^{\frac{t}{2}} b_1(\tau)^{-1}b_1(t)^{-1}(1+B_1(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-1}(1+B_1(\tau,0))^{-\frac{n}{2m}(p-1)\left(\frac{1+r_2}{1+r_1}\right)}d\tau.\\
&\lesssim \|(u,v)\|^p_{X(t)}b_1(t)^{-1}(1+B_1(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-1}\int_0^{\frac{t}{2}} b_1(\tau)^{-1}(1+B_1(\tau,0))^{-\frac{n}{2m}(\tilde{p}_{r_1,r_2}-1)}d\tau\\
&\lesssim \|(u,v)\|^p_{X(t)}b_1(t)^{-1}(1+B_1(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-1},
\end{aligned}$$

where we used $\tilde{p}_{r_1,r_2} > p_{Fuj,m}(n)$.
Now, for the integral over $[\frac{t}{2},t]$ we use (3.10) and (1.7) for $m=2$. Then we get

$$\begin{aligned}
&\|(u,v)\|^p_{X(t)} \int_{\frac{t}{2}}^t b_1(\tau)^{-1} b_1(t)^{-1}(1+B_1(t,\tau))^{-1}(1+B_2(\tau,0))^{-\frac{n}{2m}p+\frac{n}{4}} d\tau \\
&\lesssim \|(u,v)\|^p_{X(t)} \int_{\frac{t}{2}}^t b_1(\tau)^{-1} b_1(t)^{-1}(1+B_1(t,\tau))^{-1}(1+B_1(\tau,0))^{\left(-\frac{n}{2m}p+\frac{n}{4}\right)\frac{1+r_2}{1+r_1}} d\tau \\
&\lesssim \|(u,v)\|^p_{X(t)} b_1(t)^{-1}(1+B_1(t,0))^{-\frac{n}{2m}(\tilde{p}_{r_1,r_2}-1)-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)\frac{1+r_2}{1+r_1}+\varepsilon} \\
&\lesssim \|(u,v)\|^p_{X(t)} b_1(t)^{-1}(1+B_1(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-1},
\end{aligned}$$

where the condition $\tilde{p}_{r_1,r_2} > p_{Fuj,m}(n)$ is used, too. Finally, we obtain

$$\left\|u_t^{nl}(t,\cdot)\right\|_{L^2(\mathbb{R}^n)} \lesssim \|(u,v)\|^p_{X(t)} b_1(t)^{-1}(1+B_1(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-1}. \tag{3.39}$$

Now we estimate $\left\|v_t^{nl}(t,\cdot)\right\|_{L^2(\mathbb{R}^n)}$. From the estimates (1.23) in Theorem 1.4 we obtain

$$\left\|v_t^{nl}(t,\cdot)\right\|_{L^2(\mathbb{R}^n)} \lesssim \int_0^t b_2(\tau)^{-1} b_2(t)^{-1}(1+B_2(t,\tau))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-1} \||u(\tau,\cdot)|^q\|_{L^m(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)} d\tau.$$

By using (3.9), (3.10) and the relation (1.6) for the integral over $[0,\frac{t}{2}]$ we have

$$\begin{aligned}
&\|(u,v)\|^q_{X(t)} \int_0^{\frac{t}{2}} b_2(\tau)^{-1} b_2(t)^{-1}(1+B_2(t,\tau))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-1}(1+B_1(\tau,0))^{-\frac{n}{2m}(q-1)} d\tau \\
&\lesssim \|(u,v)\|^q_{X(t)} \int_0^{\frac{t}{2}} b_2(\tau)^{-1} b_2(t)^{-1}(1+B_2(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-1}(1+B_2(\tau,0))^{-\frac{n}{2m}(q-1)\left(\frac{1+r_1}{1+r_2}\right)} d\tau \\
&\lesssim \|(u,v)\|^q_{X(t)} b_2(t)^{-1}(1+B_2(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-1} \\
&\qquad \times \int_0^{\frac{t}{2}} b_2(\tau)^{-1}(1+B_2(\tau,0))^{-\frac{n}{2m}(\tilde{q}_{r_1,r_2,m}-1)-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)\frac{r_2-r_1}{1+r_2}} d\tau \\
&\lesssim \|(u,v)\|^q_{X(t)} b_1(t)^{-1}(1+B_2(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-1},
\end{aligned}$$

where we used $\tilde{q}_{r_1,r_2,m} > p_{Fuj,m}(n)$.
For estimating the second integral over $[\frac{t}{2},t]$ we use (3.10) and (1.7) for $m=2$. Then we get

$$\begin{aligned}
&\|(u,v)\|^q_{X(t)} \int_{\frac{t}{2}}^t b_2(\tau)^{-1} b_2(t)^{-1}(1+B_2(t,\tau))^{-1}(1+B_1(\tau,0))^{-\frac{n}{2m}q+\frac{n}{4}} d\tau \\
&\lesssim \|(u,v)\|^q_{X(t)} \int_{\frac{t}{2}}^t b_2(\tau)^{-1} b_2(t)^{-1}(1+B_2(t,\tau))^{-1}(1+B_2(\tau,0))^{\left(-\frac{n}{2m}q+\frac{n}{4}\right)\frac{1+r_1}{1+r_2}} d\tau \\
&\lesssim \|(u,v)\|^q_{X(t)} b_2(t)^{-1}(1+B_2(t,0))^{-\frac{n}{2m}\tilde{q}_{r_1,r_2,m}+\frac{n}{4}+\varepsilon} \\
&\lesssim \|(u,v)\|^q_{X(t)} b_2(t)^{-1}(1+B_2(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-1},
\end{aligned}$$

where $\tilde{q}_{r_1,r_2} > p_{Fuj,m}(n)$ is used for an arbitrarily small positive ε. Finally, we get

$$\left\|v_t^{nl}(t,\cdot)\right\|_{L^2(\mathbb{R}^n)} \lesssim \|(u,v)\|^q_{X(t)} b_2(t)^{-1}(1+B_2(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-1}. \tag{3.40}$$

In this way we complete the proof of the first estimate (3.6). To derive the second estimate we prove immediately (3.16) to (3.19) for $s = 1$ and similarly to (3.39) and (3.40) we prove for $0 \leq \tau \leq t$ the following estimates:

$$\begin{aligned}
&\Big\|\partial_t \int_0^t E_{1,1}(t,\tau,x) *_{(x)} \big(|v(\tau,x)|^p - |\tilde{v}(\tau,x)|^p\big)\, d\tau\Big\|_{L^2(\mathbb{R}^n)}\\
&\qquad \lesssim b_1(t)^{-1}(1+B_1(t,0))^{-\frac{n}{2}(\frac{1}{m}-\frac{1}{2})-1}\\
&\qquad\qquad \times\|(u,v)-(\tilde{u},\tilde{v})\|_{X(t)}\big(\|(u,v)\|_{X(t)}^{p-1} + \|(\tilde{u},\tilde{v})\|_{X(t)}^{p-1}\big),
\end{aligned}$$

$$\begin{aligned}
&\Big\|\partial_t \int_0^t E_{2,1}(t,\tau,x) *_{(x)} \big(|u(\tau,x)|^q - |\tilde{u}(\tau,x)|^q\big)\, d\tau\Big\|_{L^2(\mathbb{R}^n)}\\
&\qquad \lesssim b_2(t)^{-1}(1+B_2(t,0))^{-\frac{n}{2}(\frac{1}{m}-\frac{1}{2})-1}\\
&\qquad\qquad \times\|(u,v)-(\tilde{u},\tilde{v})\|_{X(t)}\big(\|(u,v)\|_{X(t)}^{q-1} + \|(\tilde{u},\tilde{v})\|_{X(t)}^{q-1}\big).
\end{aligned}$$

The proof is completed. $\square$

Remark 3.4. *If we have $r_2 < r_1$ in Theorem 3.4, then we define the modified exponents $\tilde{p}$ and $\tilde{q}$ as follows:*

$$\tilde{q} := \tilde{q}_{r_1,r_2} = \frac{1+r_1}{1+r_2}(q-1)+1,$$

$$\tilde{p} := \tilde{p}_{r_1,r_2,m} = \frac{1+r_2}{1+r_1}\Big(p-\frac{m}{2}\Big)+\frac{m}{2}.$$

In the following corollary we present a generalization of the results from [4] for semilinear damped wave equations, where the data are taken from the energy space.

Corollary 3.2. *Let $n \leq 4$ and $r_1, r_2 \in (-1,1)$. The data $(u_0,u_1),(v_0,v_1)$ are assumed to belong to $\mathcal{A}_{1,1}\times\mathcal{A}_{1,1}$. Moreover, let*

$$\min\{\tilde{q};\tilde{p}\} > 1+\frac{2}{n},$$

where

$$\begin{aligned}
\tilde{q} = \tilde{q}_{r_1,r_2,1} &= \frac{1+r_1}{1+r_2}\Big(q-\frac{1}{2}\Big)+\frac{1}{2},\ \ \tilde{p} = \tilde{p}_{r_1,r_2} = \frac{1+r_2}{1+r_1}(p-1)+1 \quad \text{if} \quad r_1 < r_2,\\
\tilde{q} = \tilde{q}_{r_1,r_2} &= \frac{1+r_1}{1+r_2}(q-1)+1,\ \ \tilde{p} = \tilde{p}_{r_1,r_2,1} = \frac{1+r_2}{1+r_1}\Big(p-\frac{1}{2}\Big)+\frac{1}{2} \quad \text{if} \quad r_2 < r_1.
\end{aligned}$$

The exponents of power nonlinearities satisfy

$$\begin{aligned}
2 \leq p,q \quad &\text{if} \quad n \leq 2,\\
2 \leq p,q \leq 3 \quad &\text{if} \quad n = 3,\\
p = q = 2 \quad &\text{if} \quad n = 4.
\end{aligned}$$

Then, there exists a constant ϵ_0 such that if

$$\|(u_0,u_1)\|_{\mathcal{A}_{1,1}} + \|(v_0,v_1)\|_{\mathcal{A}_{1,1}} \leq \epsilon_0,$$

then there exists a uniquely determined global (in time) energy solution to (3.1) in

$$\Big(\mathcal{C}\big([0,\infty),H^1(\mathbb{R}^n)\big)\cap\mathcal{C}^1\big([0,\infty),L^2(\mathbb{R}^n)\big)\Big)^2.$$

Furthermore, the solution satisfies the following decay estimates:

$$\begin{aligned}
\|u(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \big(1+B_1(t,0)\big)^{-\frac{n}{4}}\big(\|(u_0,u_1)\|_{\mathcal{A}_{1,1}}+\|(v_0,v_1)\|_{\mathcal{A}_{1,1}}\big),\\
\|\nabla u(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \big(1+B_1(t,0)\big)^{-\frac{n}{4}-\frac{1}{2}}\big(\|(u_0,u_1)\|_{\mathcal{A}_{1,1}}+\|(v_0,v_1)\|_{\mathcal{A}_{1,1}}\big),\\
\|u_t(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim b_1(t)^{-1}\big(1+B_1(t,0)\big)^{-\frac{n}{4}-1}\big(\|(u_0,u_1)\|_{\mathcal{A}_{1,1}}+\|(v_0,v_1)\|_{\mathcal{A}_{1,1}}\big),\\
\|v(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \big(1+B_2(t,0)\big)^{-\frac{n}{4}}\big(\|(u_0,u_1)\|_{\mathcal{A}_{1,1}}+\|(v_0,v_1)\|_{\mathcal{A}_{1,1}}\big),\\
\|\nabla v(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \big(1+B_2(t,0)\big)^{-\frac{n}{4}-\frac{1}{2}}\big(\|(u_0,u_1)\|_{\mathcal{A}_{1,1}}+\|(v_0,v_1)\|_{\mathcal{A}_{1,1}}\big),\\
\|v_t(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim b_2(t)^{-1}\big(1+B_2(t,0)\big)^{-\frac{n}{4}-1}\big(\|(u_0,u_1)\|_{\mathcal{A}_{1,1}}+\|(v_0,v_1)\|_{\mathcal{A}_{1,1}}\big).
\end{aligned}$$

3.2.2. Only one modified exponent is above the modified Fujita exponent

Theorem 3.5. *Let $n<\frac{2m^2}{2-m}$, $n\le\frac{2m}{m-1}$, $0<r_1<r_2<1$, and $m\in[1,2)$. The data $(u_0,u_1),(v_0,v_1)$ are assumed to belong to $\mathcal{A}_{m,1}\times\mathcal{A}_{m,1}$. Moreover, let the modified exponents satisfy*

$$\tilde{p}_{r_1,r_2}<p_{Fuj,m}(n)<\tilde{q}_{r_1,r_2,m}, \tag{3.41}$$

$$\frac{n}{2}>m\left(\frac{\tilde{q}_{r_1,r_2,m}+1+\frac{m}{2}\left(\frac{r_1-r_2}{1+r_2}\right)}{\tilde{p}_{r_1,r_2}\tilde{q}_{r_1,r_2,m}-1+\frac{m}{2}(\tilde{p}-1)\left(\frac{r_1-r_2}{1+r_2}\right)}\right), \tag{3.42}$$

where

$$\tilde{q}=\tilde{q}_{r_1,r_2,m}=\frac{1+r_1}{1+r_2}\Big(q-\frac{m}{2}\Big)+\frac{m}{2},\ \ \tilde{p}=\tilde{p}_{r_1,r_2}=\frac{1+r_2}{1+r_1}(p-1)+1.$$

The exponents p and q of the power nonlinearities satisfy

$$\begin{array}{ll}\frac{2}{m}\le\min\{p;q\}\le\max\{p;q\}<\infty & \text{if}\quad n\le 2,\\ \frac{2}{m}\le\min\{p;q\}\le\max\{p;q\}\le p_{GN}(n) & \text{if}\quad n>2.\end{array} \tag{3.43}$$

Then, there exists a constant ϵ_0 such that if

$$\|(u_0,u_1)\|_{\mathcal{A}_{m,1}}+\|(v_0,v_1)\|_{\mathcal{A}_{m,1}}\le\epsilon_0,$$

then there exists a uniquely determined global (in time) energy solution to (3.1) in

$$\Big(\mathcal{C}\big([0,\infty),H^1(\mathbb{R}^n)\big)\cap\mathcal{C}^1\big([0,\infty),L^2(\mathbb{R}^n)\big)\Big)^2.$$

Furthermore, the solution satisfies the following estimates:

$$\begin{aligned}
\|u(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \big(1+B_1(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)+\gamma_{n,m}(\tilde{p}_{r_1,r_2})} \\
&\qquad\qquad \times\big(\|(u_0,u_1)\|_{\mathcal{A}_{m,1}}+\|(v_0,v_1)\|_{\mathcal{A}_{m,1}}\big), \\
\|\nabla u(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \big(1+B_1(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{1}{2}+\gamma_{n,m}(\tilde{p}_{r_1,r_2})} \\
&\qquad\qquad \times\big(\|(u_0,u_1)\|_{\mathcal{A}_{m,1}}+\|(v_0,v_1)\|_{\mathcal{A}_{m,1}}\big), \\
\|u_t(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim b_1(t)^{-1}\big(1+B_1(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-1+\gamma_{n,m}(\tilde{p}_{r_1,r_2})} \\
&\qquad\qquad \times\big(\|(u_0,u_1)\|_{\mathcal{A}_{m,1}}+\|(v_0,v_1)\|_{\mathcal{A}_{m,1}}\big), \\
\|v(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \big(1+B_2(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)}\big(\|(u_0,u_1)\|_{\mathcal{A}_{m,1}}+\|(v_0,v_1)\|_{\mathcal{A}_{m,1}}\big), \\
\|\nabla v(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \big(1+B_2(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{1}{2}}\big(\|(u_0,u_1)\|_{\mathcal{A}_{m,1}}+\|(v_0,v_1)\|_{\mathcal{A}_{m,1}}\big). \\
\|v_t(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim b_2(t)^{-1}\big(1+B_2(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-1}\big(\|(u_0,u_1)\|_{\mathcal{A}_{m,1}}+\|(v_0,v_1)\|_{\mathcal{A}_{m,1}}\big),
\end{aligned}$$

where

$$\gamma_{n,m}(\tilde{p}_{r_1,r_2}) = -\frac{n}{2m}(\tilde{p}-1)+1$$

represent the loss of decay in comparison with the corresponding decay estimates for the solution u of the linear Cauchy problem with vanishing right hand-side.

Remark 3.5. *If we assume $r_1 = r_2$, then $\tilde{p}_{r_1,r_2} = p$ and $\tilde{q}_{r_1,r_2,m} = q$. In this case the condition (3.42) coincides with the classical condition (2.2) for weakly coupled systems having the same time-dependent coefficients in the dissipation terms.*

Proof. To prove this theorem we follow the same steps of the proof to Theorem 3.2 with modified norms for the elements of the solution space $X(t)$. Here the solution space is taken with $s=1$ and its norm contains additional terms related to suitable norms of u_t and v_t, which are generated by the new regularity of data. Namely, we define

$$X(t) = \big\{(u,v)\in\big(\mathcal{C}\big([0,t],H^1(\mathbb{R}^n)\big)\cap\mathcal{C}^1\big([0,t],L^2(\mathbb{R}^n)\big)\big)^2\big\}$$

with the norm

$$\|(u,v)\|_{X(t)} = \sup_{\tau\in[0,t]}\big\{(1+B_1(\tau,0))^{-\gamma_{n,m}(\tilde{p}_{r_1,r_2})}M_1(\tau,u)+M_2(\tau,v)\big\},$$

where

$$\begin{aligned}
M_1(\tau,u) &= (1+B_1(\tau,0))^{\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)}\|u(\tau,\cdot)\|_{L^2(\mathbb{R}^n)} \\
&\qquad +(1+B_1(\tau,0))^{\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)+\frac{1}{2}}\|\nabla u(\tau,\cdot)\|_{L^2(\mathbb{R}^n)} \\
&\qquad +b_1(\tau)(1+B_1(\tau,0))^{\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)+1}\|u_t(\tau,\cdot)\|_{L^2(\mathbb{R}^n)}, \\
M_2(\tau,v) &= (1+B_2(\tau,0))^{\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)}\|v(\tau,\cdot)\|_{L^2(\mathbb{R}^n)} \\
&\qquad +(1+B_2(\tau,0))^{\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)+\frac{1}{2}}\|\nabla v(\tau,\cdot)\|_{L^2(\mathbb{R}^n)} \\
&\qquad +b_2(\tau)(1+B_2(\tau,0))^{\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)+1}\|v_t(\tau,\cdot)\|_{L^2(\mathbb{R}^n)}.
\end{aligned}$$

Our aim is to prove the following estimates:

$$\|N(u,v)\|_{X(t)} \lesssim \|(u_0,u_1)\|_{\mathcal{A}_{m,1}} + \|(v_0,v_1)\|_{\mathcal{A}_{m,1}} + \|(u,v)\|^p_{X(t)} + \|(u,v)\|^q_{X(t)}, \quad (3.44)$$

$$\begin{aligned}\|N(u,v)-N(\tilde{u},\tilde{v})\|_{X(t)} &\lesssim \|(u,v)-(\tilde{u},\tilde{v})\|_{X(t)}\\ &\times\big(\|(u,v)\|^{p-1}_{X(t)} + \|(\tilde{u},\tilde{v})\|^{p-1}_{X(t)} + \|(u,v)\|^{q-1}_{X(t)} + \|(\tilde{u},\tilde{v})\|^{q-1}_{X(t)}\big).\end{aligned} \quad (3.45)$$

From the proof of Theorem 3.2 we may conclude (3.26) to (3.30) for $s=1$. Then it remains to prove the estimates for $\big\|u_t^{nl}(t,\cdot)\big\|_{L^2(\mathbb{R}^n)}$ and $\big\|v_t^{nl}(t,\cdot)\big\|_{L^2(\mathbb{R}^n)}$.
For the first norm we use the estimate (1.23) from Theorem 1.4, with $m=2$ for the integral over $[\frac{t}{2},t]$, and obtain

$$\begin{aligned}
&\big\|u_t^{nl}(t,\cdot)\big\|_{L^2(\mathbb{R}^n)}\\
&\lesssim \int_0^t b_1(\tau)^{-1}b_1(t)^{-1}(1+B_1(t,\tau))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-1}\||v(\tau,x)|^p\|_{L^m(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)}d\tau\\
&\lesssim \|(u,v)\|^p_{X(t)}\int_0^t b_1(\tau)^{-1}b_1(t)^{-1}(1+B_1(t,\tau))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-1}(1+B_2(t,\tau))^{-\frac{n}{2m}(p-1)}d\tau\\
&\lesssim \|(u,v)\|^p_{X(t)}\int_0^t b_1(\tau)^{-1}b_1(t)^{-1}(1+B_1(t,\tau))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-1}(1+B_1(t,\tau))^{-\frac{n}{2m}(p-1)\left(\frac{1+r_2}{1+r_1}\right)}d\tau\\
&\lesssim \|(u,v)\|^p_{X(t)}b_1(t)^{-1}(1+B_1(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-1}\int_0^{\frac{t}{2}} b_1(\tau)^{-1}(1+B_1(\tau,0))^{-\frac{n}{2m}(\tilde{p}_{r_1,r_2}-1)}d\tau\\
&\quad+\|(u,v)\|^p_{X(t)}b_1(t)^{-1}(1+B_1(t,0))^{-\frac{n}{2m}(\tilde{p}_{r_1,r_2}-1)-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)\frac{1+r_2}{1+r_1}+\varepsilon}\\
&\lesssim \|(u,v)\|^p_{X(t)}b_1(t)^{-1}(1+B_1(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-1+\gamma_{n,m}(\tilde{p}_{r_1,r_2})}.
\end{aligned}$$

Then,

$$\big\|u_t^{nl}(t,\cdot)\big\|_{L^2(\mathbb{R}^n)} \lesssim \|(u,v)\|^p_{X(t)}b_1(t)^{-1}(1+B_1(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-1+\gamma_{n,m}(\tilde{p}_{r_1,r_2})}. \quad (3.46)$$

To estimate the second norm $\big\|v_t^{nl}(t,\cdot)\big\|_{L^2(\mathbb{R}^n)}$ we take into consideration

$$\begin{aligned}
&\big\|v_t^{nl}(t,\cdot)\big\|_{L^2(\mathbb{R}^n)}\\
&\lesssim \int_0^t b_2(\tau)^{-1}b_2(t)^{-1}(1+B_2(t,\tau))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-1}\||u(\tau,\cdot)|^q\|_{L^m(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)}d\tau\\
&\lesssim \|(u,v)\|^q_{X(t)}b_2(t)^{-1}(1+B_2(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-1}\\
&\qquad\times\int_0^{\frac{t}{2}} b_2(\tau)^{-1}(1+B_2(\tau,0))^{-\frac{n}{2m}(\tilde{q}_{r_1,r_2,m}-1)-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)\frac{r_2-r_1}{1+r_2}+\gamma_{n,m}(\tilde{p}_{r_1,r_2})q\left(\frac{1+r_1}{1+r_2}\right)}d\tau\\
&\quad+\|(u,v)\|^q_{X(t)}\int_{\frac{t}{2}}^t b_2(\tau)^{-1}b_2(t)^{-1}(1+B_2(t,\tau))^{-1}\\
&\qquad\qquad\times(1+B_2(\tau,0))^{-\frac{n}{2m}\tilde{q}+\frac{n}{4}+\gamma_{n,m}(\tilde{p}_{r_1,r_2})q\left(\frac{1+r_1}{1+r_2}\right)}d\tau\\
&\lesssim \|(u,v)\|^q_{X(t)}b_2(t)^{-1}(1+B_2(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-1},
\end{aligned}$$

where we used

$$-\frac{n}{2m}(\tilde{q}_{r_1,r_2,m}-1)+\gamma_{n,m}(\tilde{p}_{r_1,r_2})q\Big(\frac{1+r_1}{1+r_2}\Big)+1+\varepsilon<0$$

which is (compare with (3.28)) equivalent to (3.42) for an arbitrarily small positive ε. Then,

$$\big\|v_t^{nl}(t,\cdot)\big\|_{L^2(\mathbb{R}^n)} \lesssim \|(u,v)\|^q_{X(t)} b_2(t)^{-1}(1+B_2(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-1}. \tag{3.47}$$

Finally, from (3.46) and (3.47) together with (3.26) to (3.30) for $s=1$ we conclude (3.44).
To prove (3.45), we conclude the estimates (3.32) to (3.35) for $s=1$ and similarly to (3.46) and (3.47) we prove

$$\begin{aligned}
&\Big\|\partial_t \int_0^t E_{1,1}(t,\tau,x) *_{(x)} \big(|v(\tau,x)|^p - |\tilde{v}(\tau,x)|^p\big)\, d\tau\Big\|_{L^2(\mathbb{R}^n)} \\
&\qquad \lesssim b_1(t)^{-1}(1+B_1(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-1+\gamma_{n,m}(\tilde{p}_{r_1,r_2})}\|(u,v)-(\tilde{u},\tilde{v})\|_{X(t)} \\
&\qquad\qquad \times\big(\|(u,v)\|^{p-1}_{X(t)} + \|(\tilde{u},\tilde{v})\|^{p-1}_{X(t)}\big),\\
&\Big\|\partial_t \int_0^t E_{2,1}(t,\tau,x) *_{(x)} \big(|u(\tau,x)|^q - |\tilde{u}(\tau,x)|^q\big)\, d\tau\Big\|_{L^2(\mathbb{R}^n)} \\
&\qquad \lesssim b_2(t)^{-1}(1+B_2(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-1}\|(u,v)-(\tilde{u},\tilde{v})\|_{X(t)} \\
&\qquad\qquad \times\big(\|(u,v)\|^{q-1}_{X(t)} + \|(\tilde{u},\tilde{v})\|^{q-1}_{X(t)}\big).
\end{aligned}$$

The proof is completed. □

If $\tilde{q}_{r_1,r_2,m} < \tilde{p}_{r_1,r_2}$, then we get the following result analogous to Theorem 3.5.

Theorem 3.6. *Let $n < \frac{2m^2}{2-m}$, $n \le \frac{2m}{m-1}$, $0 < r_1 < r_2 < 1$, and $m \in [1,2)$. The data $(u_0,u_1),(v_0,v_1)$ are assumed to belong to $\mathcal{A}_{m,1}\times\mathcal{A}_{m,1}$. Moreover, let the modified exponents satisfy*

$$\tilde{q}_{r_1,r_2,m} < p_{Fuj,m}(n) < \tilde{p}_{r_1,r_2}, \tag{3.48}$$

$$\frac{n}{2} > m\Big(\frac{\tilde{p}_{r_1,r_2} + \frac{1+r_2}{1+r_1}}{\tilde{p}_{r_1,r_2}\tilde{q}_{r_1,r_2,m} - \frac{1+r_2}{1+r_1} + \tilde{q}\big(\frac{r_2-r_1}{1+r_1}\big)}\Big), \tag{3.49}$$

where

$$\tilde{q} = \tilde{q}_{r_1,r_2,m} = \frac{1+r_1}{1+r_2}\Big(q - \frac{m}{2}\Big) + \frac{m}{2},\ \ \tilde{p} = \tilde{p}_{r_1,r_2} = \frac{1+r_2}{1+r_1}(p-1)+1.$$

The exponents p and q of the power nonlinearities satisfy the conditions

$$\begin{array}{ll} \frac{2}{m} \le \min\{p;q\} \le \max\{p;q\} < \infty & \text{if} \quad n \le 2,\\ \frac{2}{m} \le \min\{p;q\} \le \max\{p;q\} \le p_{GN}(n) & \text{if} \quad n > 2. \end{array} \tag{3.50}$$

Then, there exists a constant ϵ_0 such that if

$$\|(u_0,u_1)\|_{\mathcal{A}_{m,1}} + \|(v_0,v_1)\|_{\mathcal{A}_{m,1}} \le \epsilon_0,$$

then there exists a uniquely determined global (in time) energy solution to (3.1) in

$$\Big(\mathcal{C}\big([0,\infty),H^1(\mathbb{R}^n)\big)\cap\mathcal{C}^1\big([0,\infty),L^2(\mathbb{R}^n)\big)\Big)^2.$$

Furthermore, the solution satisfies the following estimates:

$$
\begin{aligned}
\|u(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \big(1+B_1(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)}\big(\|(u_0,u_1)\|_{\mathcal{A}_{m,1}}+\|(v_0,v_1)\|_{\mathcal{A}_{m,1}}\big),\\
\|\nabla u(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \big(1+B_1(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{1}{2}}\big(\|(u_0,u_1)\|_{\mathcal{A}_{m,1}}+\|(v_0,v_1)\|_{\mathcal{A}_{m,1}}\big),\\
\|u_t(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim b_1(t)^{-1}\big(1+B_1(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-1}\big(\|(u_0,u_1)\|_{\mathcal{A}_{m,1}}+\|(v_0,v_1)\|_{\mathcal{A}_{m,1}}\big),\\
\|v(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \big(1+B_2(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)+\gamma_{n,m}(\tilde{q}_{r_1,r_2,m})}\\
&\qquad\qquad\qquad\times\big(\|(u_0,u_1)\|_{\mathcal{A}_{m,1}}+\|(v_0,v_1)\|_{\mathcal{A}_{m,1}}\big),\\
\|\nabla v(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \big(1+B_2(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{1}{2}+\gamma_{n,m}(\tilde{q}_{r_1,r_2,m})}\\
&\qquad\qquad\qquad\times\big(\|(u_0,u_1)\|_{\mathcal{A}_{m,1}}+\|(v_0,v_1)\|_{\mathcal{A}_{m,1}}\big),\\
\|v_t(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim b_2(t)^{-1}\big(1+B_2(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-1+\gamma_{n,m}(\tilde{q}_{r_1,r_2,m})}\\
&\qquad\qquad\qquad\times\big(\|(u_0,u_1)\|_{\mathcal{A}_{m,1}}+\|(v_0,v_1)\|_{\mathcal{A}_{m,1}}\big),
\end{aligned}
$$

where

$$\gamma_{n,m}(\tilde{q}_{r_1,r_2,m}) = -\frac{n}{2m}(\tilde{q}-1)+1+\varepsilon$$

represents the loss of decay in comparison with the corresponding decay estimates for the solution v to the linear Cauchy problem with vanishing right hand-side (cf. with the loss of decay described in Theorem (2.9)).

If $r_2 < r_1$, then we change the definition of the modifies exponents $\tilde{p}$ and $\tilde{q}$. In the following two theorems we treat this case.

Theorem 3.7. *Let $n < \frac{2m^2}{2-m}$, $n \le \frac{2m}{m-1}$, $0 < r_2 < r_1 < 1$, $m \in [1,2)$. The data $(u_0,u_1),(v_0,v_1)$ are assumed to belong to $\mathcal{A}_{m,1}\times\mathcal{A}_{m,1}$. Moreover, let*

$$\tilde{p}_{r_1,r_2,m} < p_{Fuj,m}(n) < \tilde{q}_{r_1,r_2},$$

$$\frac{n}{2} > m\Big(\frac{\tilde{q}_{r_1,r_2}+\frac{1+r_1}{1+r_2}}{\tilde{p}_{r_1,r_2,m}\tilde{q}_{r_1,r_2}-\frac{1+r_1}{1+r_2}+\tilde{p}_{r_1,r_2,m}\big(\frac{r_1-r_2}{1+r_2}\big)}\Big),$$

where

$$\tilde{q}=\tilde{q}_{r_1,r_2}=\frac{1+r_1}{1+r_2}(q-1)+1,\ \ \tilde{p}=\tilde{p}_{r_1,r_2,m}=\frac{1+r_2}{1+r_1}\Big(p-\frac{m}{2}\Big)+\frac{m}{2}.$$

The exponents p and q of the power nonlinearities satisfy the conditions

$$
\begin{aligned}
\tfrac{2}{m} \le \min\{p;q\} \le \max\{p;q\} < \infty \quad &\text{if}\quad n\le 2,\\
\tfrac{2}{m} \le \min\{p;q\} \le \max\{p;q\} \le p_{GN}(n) \quad &\text{if}\quad n>2.
\end{aligned}
$$

Then, there exists a constant ϵ_0 such that if

$$\|(u_0,u_1)\|_{\mathcal{A}_{m,1}}+\|(v_0,v_1)\|_{\mathcal{A}_{m,1}}\le\epsilon_0,$$

then there exists a uniquely determined global (in time) energy solution to (3.1) in

$$\Big(\mathcal{C}\big([0,\infty),H^1(\mathbb{R}^n)\big)\cap\mathcal{C}^1\big([0,\infty),L^2(\mathbb{R}^n)\big)\Big)^2.$$

Furthermore, the solution satisfies the following estimates:

$$\begin{aligned}
\|u(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \big(1+B_1(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)+\gamma_{n,m}(\tilde{p}_{r_1,r_2,m})} \\
&\qquad\qquad \times\big(\|(u_0,u_1)\|_{\mathcal{A}_{m,1}}+\|(v_0,v_1)\|_{\mathcal{A}_{m,1}}\big),\\
\|\nabla u(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \big(1+B_1(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{1}{2}+\gamma_{n,m}(\tilde{p}_{r_1,r_2,m})} \\
&\qquad\qquad \times\big(\|(u_0,u_1)\|_{\mathcal{A}_{m,1}}+\|(v_0,v_1)\|_{\mathcal{A}_{m,1}}\big),\\
\|u_t(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim b_1(t)^{-1}\big(1+B_1(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-1+\gamma_{n,m}(\tilde{p}_{r_1,r_2,m})} \\
&\qquad\qquad \times\big(\|(u_0,u_1)\|_{\mathcal{A}_{m,1}}+\|(v_0,v_1)\|_{\mathcal{A}_{m,1}}\big),\\
\|v(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \big(1+B_2(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)}\big(\|(u_0,u_1)\|_{\mathcal{A}_{m,1}}+\|(v_0,v_1)\|_{\mathcal{A}_{m,1}}\big),\\
\|\nabla v(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \big(1+B_2(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{1}{2}}\big(\|(u_0,u_1)\|_{\mathcal{A}_{m,1}}+\|(v_0,v_1)\|_{\mathcal{A}_{m,1}}\big).\\
\|v_t(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim b_2(t)^{-1}\big(1+B_2(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-1}\big(\|(u_0,u_1)\|_{\mathcal{A}_{m,1}}+\|(v_0,v_1)\|_{\mathcal{A}_{m,1}}\big),
\end{aligned}$$

where

$$\gamma_{n,m}(\tilde{p}_{r_1,r_2,m}) = -\frac{n}{2m}(\tilde{p}-1)+1+\varepsilon$$

represents the loss of decay of solutions in comparison with the corresponding decay estimates for the solution u to the linear Cauchy problem with vanishing right hand-side (cf. with the loss of decay described in Theorem (2.9)).

Theorem 3.8. *Let $n<\frac{2m^2}{2-m}$, $n\leq\frac{2m}{m-1}$, $0<r_2<r_1<1$, and $m\in[1,2)$. The data $(u_0,u_1),(v_0,v_1)$ are assumed to belong to $\mathcal{A}_{m,1}\times\mathcal{A}_{m,1}$. Moreover, let*

$$\tilde{q}_{r_1,r_2}<p_{Fuj,m}(n)<\tilde{p}_{r_1,r_2,m},$$

$$\frac{n}{2}>m\Big(\frac{\tilde{p}_{r_1,r_2,m}+1+\frac{m}{2}\frac{r_2-r_1}{1+r_1}}{\tilde{p}_{r_1,r_2,m}\tilde{q}_{r_1,r_2}-1+\frac{m}{2}(\tilde{q}_{r_1,r_2}-1)\frac{r_2-r_1}{1+r_1}}\Big),$$

where

$$\tilde{q}=\tilde{q}_{r_1,r_2}=\frac{1+r_1}{1+r_2}(q-1)+1,\ \ \tilde{p}=\tilde{p}_{r_1,r_2,m}=\frac{1+r_2}{1+r_1}\Big(p-\frac{m}{2}\Big)+\frac{m}{2}.$$

The exponents p and q of the power nonlinearities satisfy the conditions

$$\begin{array}{ll}
\frac{2}{m}\leq\min\{p;q\}\leq\max\{p;q\}<\infty & \text{if}\quad n\leq 2,\\
\frac{2}{m}\leq\min\{p;q\}\leq\max\{p;q\}\leq p_{GN}(n) & \text{if}\quad n>2.
\end{array}$$

Then, there exists a constant ϵ_0 such that if

$$\|(u_0,u_1)\|_{\mathcal{A}_{m,1}}+\|(v_0,v_1)\|_{\mathcal{A}_{m,1}}\leq\epsilon_0,$$

then there exists a uniquely determined global (in time) energy solution to (3.1) in

$$\Big(\mathcal{C}\big([0,\infty),H^1(\mathbb{R}^n)\big)\cap\mathcal{C}^1\big([0,\infty),L^2(\mathbb{R}^n)\big)\Big)^2.$$

Furthermore, the solution satisfies the following estimates:

$$\|u(t,\cdot)\|_{L^2(\mathbb{R}^n)}\lesssim\big(1+B_1(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)}\big(\|(u_0,u_1)\|_{\mathcal{A}_{m,1}}+\|(v_0,v_1)\|_{\mathcal{A}_{m,1}}\big),$$

$$\|\nabla u(t,\cdot)\|_{L^2(\mathbb{R}^n)} \lesssim \big(1+B_1(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{1}{2}}\big(\|(u_0,u_1)\|_{\mathcal{A}_{m,1}}+\|(v_0,v_1)\|_{\mathcal{A}_{m,1}}\big),$$

$$\|u_t(t,\cdot)\|_{L^2(\mathbb{R}^n)} \lesssim b_1(t)^{-1}\big(1+B_1(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-1}\big(\|(u_0,u_1)\|_{\mathcal{A}_{m,1}}+\|(v_0,v_1)\|_{\mathcal{A}_{m,1}}\big),$$

$$\|v(t,\cdot)\|_{L^2(\mathbb{R}^n)} \lesssim \big(1+B_2(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)+\gamma_{n,m}(\tilde{q}_{r_1,r_2})}\big(\|(u_0,u_1)\|_{\mathcal{A}_{m,1}}+\|(v_0,v_1)\|_{\mathcal{A}_{m,1}}\big),$$

$$\|\nabla v(t,\cdot)\|_{L^2(\mathbb{R}^n)} \lesssim \big(1+B_2(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{1}{2}+\gamma_{n,m}(\tilde{q}_{r_1,r_2})}\big(\|(u_0,u_1)\|_{\mathcal{A}_{m,1}}+\|(v_0,v_1)\|_{\mathcal{A}_{m,1}}\big).$$

$$\|v_t(t,\cdot)\|_{L^2(\mathbb{R}^n)} \lesssim b_2(t)^{-1}\big(1+B_2(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-1+\gamma_{n,m}(\tilde{q}_{r_1,r_2})}\big(\|(u_0,u_1)\|_{\mathcal{A}_{m,1}}+\|(v_0,v_1)\|_{\mathcal{A}_{m,1}}\big),$$

where

$$\gamma_{n,m}(\tilde{q}_{r_1,r_2}) = -\frac{n}{2m}(\tilde{q}-1)+1$$

represents the loss of decay of solutions in comparison with the corresponding decay estimates for the solution v to the linear Cauchy problem with vanishing right hand-side.

We summarize the results of Theorems 3.5 to 3.8 in the following table:

	$\tilde{p} < p_{Fuj,m}(n) < \tilde{q}$	$\tilde{q} < p_{Fuj,m}(n) < \tilde{p}$
$r_1 < r_2$ $\tilde{q} = \tilde{q}_{r_1,r_2,m}$ $\tilde{p} = \tilde{p}_{r_1,r_2}$	Loss of decay in the estimate for u $\gamma_{n,m}(\tilde{p}) = -\frac{n}{2m}(\tilde{p}-1)+1$ interaction condition $\frac{n}{2} > m\Big(\frac{\tilde{q}+1+\frac{m}{2}\left(\frac{r_1-r_2}{1+r_2}\right)}{\tilde{p}\tilde{q}-1+\frac{m}{2}(\tilde{p}-1)\left(\frac{r_1-r_2}{1+r_2}\right)}\Big)$	Loss of decay in the estimate for v $\gamma_{n,m}(\tilde{q}) = -\frac{n}{2m}(\tilde{q}-1)+1+\varepsilon$ interaction condition $\frac{n}{2} > m\Big(\frac{\tilde{p}+\frac{1+r_2}{1+r_1}}{\tilde{p}\tilde{q}-\frac{1+r_2}{1+r_1}+\tilde{q}\left(\frac{r_2-r_1}{1+r_1}\right)}\Big)$
$r_2 < r_1$ $\tilde{q} = \tilde{q}_{r_1,r_2}$ $\tilde{p} = \tilde{p}_{r_1,r_2,m}$	Loss of decay in the estimate for u $\gamma_{n,m}(\tilde{p}) = -\frac{n}{2m}(\tilde{p}-1)+1+\varepsilon$ interaction condition $\frac{n}{2} > m\Big(\frac{\tilde{q}+\frac{1+r_1}{1+r_2}}{\tilde{p}\tilde{q}-\frac{1+r_1}{1+r_2}+\tilde{p}\left(\frac{r_1-r_2}{1+r_2}\right)}\Big)$	Loss of decay in the estimate for v $\gamma_{n,m}(\tilde{q}) = -\frac{n}{2m}(\tilde{q}-1)+1$ interaction condition $\frac{n}{2} > m\Big(\frac{\tilde{p}+1+\frac{m}{2}\left(\frac{r_2-r_1}{1+r_1}\right)}{\tilde{p}\tilde{q}-1+\frac{m}{2}(\tilde{q}-1)\left(\frac{r_2-r_1}{1+r_1}\right)}\Big)$

Example 3.4. *Let us choose $n=2$ in Theorem 3.5. If we choose the additional regularity $m=\frac{7}{4}$, then we obtain $\frac{2}{m}=\frac{8}{7}$ and $p_{Fuj,\frac{7}{4}}(2)=\frac{11}{4}$. Finally, for $p=\frac{3}{2}\in\left[\frac{8}{7},\infty\right)$, $q=20\in\left[\frac{8}{7},\infty\right)$ and $\frac{1+r_1}{1+r_2}=\frac{1}{2}$ we get*

$$\tilde{p} = 2 < p_{Fuj,\frac{7}{4}}(2) < \frac{167}{16} = \tilde{q}.$$

Moreover, the modified exponents $\tilde{p}$ and $\tilde{q}$ satisfy the condition (3.42).

3.3. Data from Sobolev spaces with suitable regularity

Theorem 3.9. *Let $n \geq 4$, $s_1, s_2 \in [3, \frac{n}{2}+1]$, $0 < s_2 - s_1 < 1$, $\lceil s_1 \rceil \neq \lceil s_2 \rceil$, and $-1 < r_1 < r_2 < 1$. The data $(u_0,u_1),(v_0,v_1)$ are supposed to belong to $\mathcal{A}_{m,s_1}\times\mathcal{A}_{m,s_2}$, for $m\in[1,2)$. Furthermore, let*

$$\tilde{q} > \frac{2m}{n}\left(\frac{s_2+1}{2}\right)+1, \tag{3.51}$$

where $\tilde{q} = \tilde{q}_{r_1,r_2,m} = \frac{1+r_1}{1+r_2}\big(q-\frac{m}{2}\big)+\frac{m}{2}$. The exponents p and q of the power nonlinearities satisfy the conditions

$$\begin{array}{llll}
\lceil s_1\rceil < p, & \lceil s_2\rceil < q & \text{if} & n \leq 2s_1,\\
\lceil s_1\rceil < p, & \lceil s_2\rceil < q \leq 1+\frac{2}{n-2s_1} & \text{if} & 2s_1 < n \leq 2s_2,\\
\lceil s_1\rceil < p \leq 1+\frac{2}{n-2s_2}, & \lceil s_2\rceil < q \leq 1+\frac{2}{n-2s_1} & \text{if} & n > 2s_2.
\end{array} \tag{3.52}$$

Then, there exists a constant ϵ_0 such that if

$$\|(u_0,u_1)\|_{\mathcal{A}_{m,s_1}} + \|(v_0,v_1)\|_{\mathcal{A}_{m,s_2}} \leq \epsilon_0,$$

then there exists a uniquely determined globally (in time) energy solution to (3.1) in

$$\Big(\mathcal{C}\big([0,\infty),H^{s_1}(\mathbb{R}^n)\big)\cap\mathcal{C}^1\big([0,\infty),H^{s_1-1}(\mathbb{R}^n)\big)\Big)$$
$$\times\Big(\mathcal{C}\big([0,\infty),H^{s_2}(\mathbb{R}^n)\big)\cap\mathcal{C}^1\big([0,\infty),H^{s_2-1}(\mathbb{R}^n)\big)\Big).$$

Furthermore, the solution satisfies the estimates

$$\begin{aligned}
\|u(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \big(1+B_1(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)}\left(\|(u_0,u_1)\|_{\mathcal{A}_{m,s_1}}+\|(v_0,v_1)\|_{\mathcal{A}_{m,s_2}}\right),\\
\||D|^{s_1}u(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \big(1+B_1(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s_1}{2}}\\
&\qquad\times\left(\|(u_0,u_1)\|_{\mathcal{A}_{m,s_1}}+\|(v_0,v_1)\|_{\mathcal{A}_{m,s_2}}\right),\\
\|u_t(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim b_1(t)^{-1}\big(1+B_1(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-1}\\
&\qquad\times\left(\|(u_0,u_1)\|_{\mathcal{A}_{m,s_1}}+\|(v_0,v_1)\|_{\mathcal{A}_{m,s_2}}\right),\\
\||D|^{s_1-1}u_t(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim b_1(t)^{-1}\big(1+B_1(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-1-\frac{s_1-1}{2}}\\
&\qquad\times\left(\|(u_0,u_1)\|_{\mathcal{A}_{m,s_1}}+\|(v_0,v_1)\|_{\mathcal{A}_{m,s_2}}\right),\\
\|v(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \big(1+B_2(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)}\left(\|(u_0,u_1)\|_{\mathcal{A}_{m,s_1}}+\|(v_0,v_1)\|_{\mathcal{A}_{m,s_2}}\right),\\
\||D|^{s_2}v(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \big(1+B_2(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s_2}{2}}\\
&\qquad\times\left(\|(u_0,u_1)\|_{\mathcal{A}_{m,s_1}}+\|(v_0,v_1)\|_{\mathcal{A}_{m,s_2}}\right),\\
\|v_t(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim b_2(t)^{-1}\big(1+B_2(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-1}\\
&\qquad\left(\|(u_0,u_1)\|_{\mathcal{A}_{m,s_1}}+\|(v_0,v_1)\|_{\mathcal{A}_{m,s_2}}\right),\\
\||D|^{s_2-1}v_t(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim b_2(t)^{-1}\big(1+B_2(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-1-\frac{s_2-1}{2}}\\
&\qquad\times\left(\|(u_0,u_1)\|_{\mathcal{A}_{m,s_1}}+\|(v_0,v_1)\|_{\mathcal{A}_{m,s_2}}\right).
\end{aligned}$$

Remark 3.6. *Remark 2.13 remains valid for the case which is treated in Theorem 3.9, where the coefficients of the dissipation terms are supposed to be different.*

Remark 3.7. *If we suppose in Theorem 3.9 the assumption $-1 < r_2 < r_1 < 1$, then we replace condition (3.51) by the following condition:*

$$\tilde{p} > \frac{2m}{n}\Big(\frac{s_1+1}{2}\Big)+1,$$

where $\tilde{p} = \tilde{p}_{r_1,r_2,m} = \frac{1+r_1}{1+r_2}\big(p-\frac{m}{2}\big)+\frac{m}{2}$.

Proof. Let us define the space of solutions $X(t)$ by

$$\begin{aligned} X(t) &= \Big\{(u,v) \in \big[\mathcal{C}([0,t],H^{s_1}(\mathbb{R}^n)) \cap \mathcal{C}^1([0,t],H^{s_1-1}(\mathbb{R}^n))\big] \\ &\qquad \times \big[\mathcal{C}([0,t],H^{s_2}(\mathbb{R}^n)) \cap \mathcal{C}^1([0,t],H^{s_2-1}(\mathbb{R}^n))\big]\Big\} \end{aligned}$$

with the norm

$$\|(u,v)\|_{X(t)} = \sup_{\tau\in[0,t]} \{M_1(\tau,u) + M_2(\tau,v)\},$$

where

$$\begin{aligned} M_1(\tau,u) &= \big(1+B_1(\tau,0)\big)^{\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)}\|u(\tau,\cdot)\|_{L^2(\mathbb{R}^n)} \\ &\quad + b_1(\tau)\big(1+B_1(\tau,0)\big)^{\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)+1}\|u_t(\tau,\cdot)\|_{L^2(\mathbb{R}^n)} \\ &\quad + b_1(\tau)\big(1+B_1(\tau,0)\big)^{\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)+\frac{s_1-1}{2}+1}\||D|^{s_1-1}u_t(\tau,\cdot)\|_{L^2(\mathbb{R}^n)} \\ &\quad + \big(1+B_1(\tau,0)\big)^{\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)+\frac{s_1}{2}}\||D|^{s_1}u(\tau,\cdot)\|_{L^2(\mathbb{R}^n)}, \end{aligned}$$

and

$$\begin{aligned} M_2(\tau,v) &= \big(1+B_2(\tau,0)\big)^{\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)}\|v(\tau,\cdot)\|_{L^2(\mathbb{R}^n)} \\ &\quad + b_2(\tau)\big(1+B_2(\tau,0)\big)^{\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)+1}\|v_t(\tau,\cdot)\|_{L^2(\mathbb{R}^n)} \\ &\quad + b_2(\tau)\big(1+B_2(\tau,0)\big)^{\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)+\frac{s_2-1}{2}+1}\||D|^{s_2-1}v_t(\tau,\cdot)\|_{L^2(\mathbb{R}^n)} \\ &\quad + \big(1+B_2(\tau,0)\big)^{\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)+\frac{s_2}{2}}\||D|^{s_2}v(\tau,\cdot)\|_{L^2(\mathbb{R}^n)}. \end{aligned}$$

Let N be the mapping as defined in (3.5). Then our aim is to prove

$$\|N(u,v)\|_{X(t)} \lesssim \|(u_0,u_1)\|_{\mathcal{A}_{m,s_1}} + \|(v_0,v_1)\|_{\mathcal{A}_{m,s_2}} + \|(u,v)\|^p_{X(t)} + \|(u,v)\|^q_{X(t)}, \tag{3.53}$$

$$\begin{aligned} \|N(u,v)-N(\tilde{u},\tilde{v})\|_{X(t)} &\lesssim \|(u,v)-(\tilde{u},\tilde{v})\|_{X(t)} \\ &\quad \times\big(\|(u,v)\|^{p-1}_{X(t)} + \|(\tilde{u},\tilde{v})\|^{p-1}_{X(t)} + \|(u,v)\|^{q-1}_{X(t)} + \|(\tilde{u},\tilde{v})\|^{q-1}_{X(t)}\big). \end{aligned} \tag{3.54}$$

We can immediately conclude the estimate of the linear part of the operator N as follows:

$$\|(u^{ln},v^{ln})\|_{X(t)} \lesssim \|(u_0,u_1)\|_{\mathcal{A}_{m,s_1}} + \|(v_0,v_1)\|_{\mathcal{A}_{m,s_2}}. \tag{3.55}$$

For the nonlinear part we only show how to estimate the norms $\||D|^{s_1-1}u_t^{nl}(t,\cdot)\|_{L^2(\mathbb{R}^n)}$ and $\||D|^{s_2-1}v_t^{nl}(t,\cdot)\|_{L^2(\mathbb{R}^n)}$. From the estimate (1.27) of Theorem 1.6 it follows

$$\begin{aligned} &\||D|^{s_1-1}u_t^{nl}(t,\cdot)\|_{L^2(\mathbb{R}^n)} \\ &\lesssim \int_0^{\frac{t}{2}} b_1(\tau)^{-1}b_1(t)^{-1}(1+B_1(t,\tau))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s_1-1}{2}-1}\||v(\tau,x)|^p\|_{L^m(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)\cap\dot{H}^{s_1-1}(\mathbb{R}^n)}d\tau \\ &\quad + \int_{\frac{t}{2}}^t b_1(\tau)^{-1}b_1(t)^{-1}(1+B_1(t,\tau))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s_1-1}{2}-1}\||v(\tau,x)|^p\|_{L^m(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)\cap\dot{H}^{s_1-1}(\mathbb{R}^n)}d\tau. \end{aligned}$$

As we did in Chapter 1, Section 2.3, after using the Gagliardo-Nirenberg inequality we get for $0 \le \tau \le t$ the estimates

$$\|v(\tau,\cdot)\|^p_{L^{mp}(\mathbb{R}^n)} \lesssim (1+B_2(\tau,0))^{-\frac{n}{2m}p+\frac{n}{2m}}\|(u,v)\|^p_{X(t)},$$

$$\|v(\tau,\cdot)\|^p_{L^{2p}(\mathbb{R}^n)} \lesssim (1+B_2(\tau,0))^{-\frac{n}{2m}p+\frac{n}{4}}\|(u,v)\|^p_{X(t)},$$

where the exponent p satisfies the following conditions which are included in (3.52):

$$\begin{array}{ll} \frac{2}{m} \leq p & \text{if} \quad n \leq 2s_2, \\ \frac{2}{m} \leq p \leq \frac{n}{n-2s_2} & \text{if} \quad n > 2s_2. \end{array}$$

To estimate $\||v(\tau,x)|^p\|_{\dot{H}^{s_1-1}(\mathbb{R}^n)}$ we use the fractional chain rule from Proposition A.4. Then for $0 \leq \tau \leq t$ we obtain

$$\||v(\tau,x)|^p\|_{\dot{H}^{s_1-1}(\mathbb{R}^n)} \lesssim (1+B_2(\tau,0))^{-\frac{n}{2m}p+\frac{n}{4}-\frac{s_1-1}{2}}\|(u,v)\|^p_{X(t)},$$

for $p > \lceil s_1-1\rceil$ and $p \leq 1+\frac{2}{n-2s_2}$ if $n > 2s_2$.
By using these estimates we get for the integral over $[0,\frac{t}{2}]$

$$\begin{aligned}
&\|(u,v)\|^p_{X(t)}\int_0^{\frac{t}{2}} b_1(\tau)^{-1}b_1(t)^{-1}(1+B_1(t,\tau))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s_1-1}{2}-1}(1+B_2(\tau,0))^{-\frac{n}{2m}p+\frac{n}{2m}}d\tau \\
&\lesssim \|(u,v)\|^p_{X(t)}b_1(t)^{-1}(1+B_1(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s_1-1}{2}-1}\int_0^{\frac{t}{2}} b_1(\tau)^{-1}(1+B_2(\tau,0))^{-\frac{n}{2m}p+\frac{n}{2m}}d\tau \\
&\lesssim \|(u,v)\|^p_{X(t)}b_1(t)^{-1}(1+B_1(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s_1-1}{2}-1} \\
&\qquad\qquad \times\int_0^{\frac{t}{2}} b_1(\tau)^{-1}(1+B_1(\tau,0))^{\frac{1+r_2}{1+r_1}\left(-\frac{n}{2m}p+\frac{n}{2m}\right)}d\tau \\
&\lesssim \|(u,v)\|^p_{X(t)}b_1(t)^{-1}(1+B_1(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s_1-1}{2}-1} \\
&\qquad\qquad \times\int_0^{\frac{t}{2}} b_1(\tau)^{-1}(1+B_1(\tau,0))^{-\frac{n}{2m}(\tilde{p}-1)}d\tau \\
&\lesssim \|(u,v)\|^p_{X(t)}b_1(t)^{-1}(1+B_1(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s_1-1}{2}-1},
\end{aligned}$$

where we used

$$\tilde{p} := \tilde{p}_{r_1,r_2} = \frac{1+r_2}{1+r_1}(p-1)+1 > p_{Fuj,m}(n)$$

from (3.52), in particular, $\tilde{p} > p > \lceil s_1\rceil > p_{Fuj,m}(n)$ for $n \geq 4$, $s_1 > 3$ and $r_1 < r_2$.
For the integral over $[\frac{t}{2},t]$ we have

$$\begin{aligned}
&\int_{\frac{t}{2}}^t b_1(\tau)^{-1}b_1(t)^{-1}(1+B_1(t,\tau))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s_1-1}{2}-1}\||v(\tau,x)|^p\|_{L^m(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)\cap\dot{H}^{s_1-1}(\mathbb{R}^n)}d\tau \\
&\lesssim \|(u,v)\|^p_{X(t)}b_1(\tau)^{-1}(1+B_1(t,0))^{-\frac{n}{2m}\tilde{p}+\frac{n}{4}}\int_{\frac{t}{2}}^t b_1(\tau)^{-1}(1+B_1(t,\tau))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s_1-1}{2}-1}d\tau \\
&\lesssim \|(u,v)\|^p_{X(t)}b_1(t)^{-1}(1+B_1(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s_1-1}{2}-1}
\end{aligned}$$

for

$$\tilde{p} > \frac{2m}{n}\Big(\frac{s_1+1}{2}\Big)+1-\Big(1-\frac{m}{2}\Big)\frac{r_2-r_1}{1+r_1}$$

which is satisfied from condition (3.52). Indeed, for $s > 3$ and $n \geq 4$ we have

$$\begin{aligned}
\frac{2m}{n}\Big(\frac{s_1+1}{2}\Big)+1-\Big(1-\frac{m}{2}\Big)\frac{r_2-r_1}{1+r_1} &< \frac{2m}{n}\Big(\frac{s_1+1}{2}\Big)+1 \\
&\leq \Big(\frac{s_1+1}{2}\Big)+1 < s_1 \leq \lceil s_1\rceil.
\end{aligned}$$

Hence,

$$\tilde{p} > \lceil s_1 \rceil \Rightarrow \tilde{p} > \frac{2m}{n}\Big(\frac{s_1+1}{2}\Big) + 1 - \Big(1-\frac{m}{2}\Big)\frac{r_2-r_1}{1+r_1}. \tag{3.56}$$

Summarizing, we obtain

$$\||D|^{s_1-1}u_t^{nl}(t,\cdot)\|_{L^2(\mathbb{R}^n)} \lesssim \|(u,v)\|_{X(t)}^p b_1(t)^{-1}(1+B_1(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s_1-1}{2}-1}. \tag{3.57}$$

In a similar way we can also derive

$$\begin{aligned}
\|u^{nl}(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \|(u,v)\|_{X(t)}^p (1+B_1(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)}, && (3.58)\\
\|u_t^{nl}(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \|(u,v)\|_{X(t)}^p b_1(t)^{-1}(1+B_1(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-1}, && (3.59)\\
\||D|^{s_1}u^{nl}(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \|(u,v)\|_{X(t)}^p (1+B_1(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s_1}{2}}. && (3.60)
\end{aligned}$$

Following the same ideas to estimate $\||D|^{s_1-1}u_t^{nl}(t,\cdot)\|_{L^2(\mathbb{R}^n)}$, one can get the following estimates:

$$\||D|^{s_2-1}v_t^{nl}(t,\cdot)\|_{L^2(\mathbb{R}^n)} \lesssim \|(u,v)\|_{X(t)}^q b_2(t)^{-1}(1+B_2(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s_2-1}{2}-1}, \tag{3.61}$$

$$\begin{aligned}
\|v^{nl}(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \|(u,v)\|_{X(t)}^q (1+B_2(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)}, && (3.62)\\
\|v_t^{nl}(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \|(u,v)\|_{X(t)}^q b_2(t)^{-1}(1+B_2(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-1}, && (3.63)\\
\||D|^{s_2}v^{nl}(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \|(u,v)\|_{X(t)}^q (1+B_2(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s_2}{2}}, && (3.64)
\end{aligned}$$

where

$$\begin{array}{ll}
\frac{2}{m} \le q & \text{if} \quad n \le 2s_1,\\
\frac{2}{m} \le q \le 1+\frac{2}{n-2s_1} & \text{if} \quad n > 2s_1,
\end{array}$$

and

$$q > \lceil s_2 - 1\rceil, \ \ s_2 - 1 < s_1, \ \ \tilde{q} > \frac{2m}{n}\Big(\frac{s_2+1}{2}\Big) + 1.$$

The last condition can not be concluded from $q > \lceil s_2 \rceil$ because of $\tilde{q} < q$.
From (3.57) to (3.64) we obtain

$$\|(u^{nl}, v^{nl})\|_{X(t)} \lesssim \|(u,v)\|_{X(t)}^p + \|(u,v)\|_{X(t)}^q. \tag{3.65}$$

Finally, the estimates (3.55) and (3.65) complete the proof of (3.53).
To prove (3.54) we assume that (u,v) and $(\tilde{u},\tilde{v})$ are two vector-functions belonging to $X(t)$. Then we have

$$\begin{aligned}
&N(u,v) - N(\tilde{u},\tilde{v})\\
&\quad = (u^{nl}(t,x) - \tilde{u}^{nl}(t,x), v^{nl}(t,x) - \tilde{v}^{nl}(t,x))\\
&\quad = \Big(\int_0^t E_{1,1}(t,\tau,x) *_{(x)} \big(|v(\tau,x)|^p - |\tilde{v}(\tau,x)|^p\big) d\tau,\\
&\qquad\quad \int_0^t E_{2,1}(t,\tau,x) *_{(x)} \big(|u(\tau,x)|^q - |\tilde{u}(\tau,x)|^q\big) d\tau\Big).
\end{aligned}$$

For the first component we use the estimates (1.27) from Theorem 1.6. Then we get

$$\begin{aligned}
&\Big\||D|^{s_1-1}\partial_t\int_0^t E_{1,1}(t,\tau,x)*_{(x)}\big(|v(\tau,x)|^p-|\tilde{v}(\tau,x)|^p\big)d\tau\Big\|_{L^2(\mathbb{R}^n)}\\
&\quad\lesssim\int_0^t b_1(t)^{-1}b_1(\tau)^{-1}(1+B_1(t,\tau))^{-\frac{n}{2}(\frac{1}{m}-\frac{1}{2})-\frac{s_1-1}{2}-1}\\
&\qquad\qquad\times\||v(\tau,x)|^p-|\tilde{v}(\tau,x)|^p\|_{L^m(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)\cap\dot{H}^{s_1-1}(\mathbb{R}^n)}d\tau.
\end{aligned}$$

Applying Hölder's inequality and Gagliardo-Nirenberg inequality as we did in the previous proofs of Sections 3.1 and 3.2 implies for $0\le\tau\le t$

$$\begin{aligned}
\||v(\tau,x)|^p-|\tilde{v}(\tau,x)|^p\|_{L^m(\mathbb{R}^n)} &\lesssim (1+B_2(\tau,0))^{-\frac{n}{2m}p+\frac{n}{2m}}\sup_{\tau\in[0,t]}M_2(\tau,v-\tilde{v})\\
&\qquad\times\big(\|(u,v)\|^{p-1}_{X(t)}+\|(\tilde{u},\tilde{v})\|^{p-1}_{X(t)}\big),\\
\||v(\tau,x)|^p-|\tilde{v}(\tau,x)|^p\|_{L^2(\mathbb{R}^n)} &\lesssim (1+B_2(\tau,0))^{-\frac{n}{2m}p+\frac{n}{4}}\sup_{\tau\in[0,t]}M_2(\tau,v-\tilde{v})\\
&\qquad\times\big(\|(u,v)\|^{p-1}_{X(t)}+\|(\tilde{u},\tilde{v})\|^{p-1}_{X(t)}\big).
\end{aligned}$$

To estimate $\||v(\tau,x)|^p-|\tilde{v}(\tau,x)|^p\|_{\dot{H}^{s_1-1}(\mathbb{R}^n)}$ we use the estimate (1.78) from the proof of Theorem 1.10. Then we obtain

$$\begin{aligned}
\||v(\tau,x)|^p-|\tilde{v}(\tau,x)|^p\|_{\dot{H}^{s_1-1}(\mathbb{R}^n)} &\lesssim (1+B_1(\tau,0))^{-\frac{n}{2m}p+\frac{n}{4}-\frac{s_1-1}{2}}\sup_{\tau\in[0,t]}M_2(\tau,v-\tilde{v})\\
&\qquad\times\big(\|(u,v)\|^{p-1}_{X(t)}+\|(\tilde{u},\tilde{v})\|^{p-1}_{X(t)}\big),
\end{aligned}$$

where $p>\lceil s_1\rceil$ and

$$\begin{array}{ll}
\frac{2}{m}\le p & \text{if } n\le 2s_2,\\
\frac{2}{m}\le p\le 1+\frac{2}{n-2s_2} & \text{if } n>2s_2.
\end{array}$$

Using the last estimates we can prove

$$\begin{aligned}
&\Big\||D|^{s_1-1}\partial_t\int_0^t E_{1,1}(t,\tau,x)*_{(x)}\big(|v(\tau,x)|^p-|\tilde{v}(\tau,x)|^p\big)d\tau\Big\|_{L^2(\mathbb{R}^n)}\\
&\qquad\lesssim b_1(t)^{-1}(1+B_1(t,0))^{-\frac{n}{2}(\frac{1}{m}-\frac{1}{2})-\frac{s_1-1}{2}-1}\|(u,v)-(\tilde{u},\tilde{v})\|_{X(t)}\\
&\qquad\qquad\times\big(\|(u,v)\|^{p-1}_{X(t)}+\|(\tilde{u},\tilde{v})\|^{p-1}_{X(t)}\big),
\end{aligned}$$

$$\begin{aligned}
&\Big\|\int_0^t E_{1,1}(t,\tau,x)*_{(x)}\big(|v(\tau,x)|^p-|\tilde{v}(\tau,x)|^p\big)d\tau\Big\|_{L^2(\mathbb{R}^n)}\\
&\qquad\lesssim (1+B_1(t,0))^{-\frac{n}{2}(\frac{1}{m}-\frac{1}{2})}\|(u,v)-(\tilde{u},\tilde{v})\|_{X(t)}\\
&\qquad\qquad\times\big(\|(u,v)\|^{p-1}_{X(t)}+\|(\tilde{u},\tilde{v})\|^{p-1}_{X(t)}\big),
\end{aligned}$$

$$\begin{aligned}
&\Big\|\partial_t\int_0^t E_{1,1}(t,\tau,x)*_{(x)}\big(|v(\tau,x)|^p-|\tilde{v}(\tau,x)|^p\big)d\tau\Big\|_{L^2(\mathbb{R}^n)}\\
&\qquad\lesssim b_1(t)^{-1}(1+B_1(t,0))^{-\frac{n}{2}(\frac{1}{m}-\frac{1}{2})-1}\|(u,v)-(\tilde{u},\tilde{v})\|_{X(t)}\\
&\qquad\qquad\times\big(\|(u,v)\|^{p-1}_{X(t)}+\|(\tilde{u},\tilde{v})\|^{p-1}_{X(t)}\big),
\end{aligned}$$

$$\Big\||D|^{s_1}\int_0^t E_{1,1}(t,\tau,x)*_{(x)}\big(|v(\tau,x)|^p-|\tilde v(\tau,x)|^p\big)d\tau\Big\|_{L^2(\mathbb{R}^n)}$$
$$\lesssim (1+B_1(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s_1}{2}}\|(u,v)-(\tilde u,\tilde v)\|_{X(t)}$$
$$\times\big(\|(u,v)\|^{p-1}_{X(t)}+\|(\tilde u,\tilde v)\|^{p-1}_{X(t)}\big).$$

Analogously, we get

$$\Big\|\int_0^t E_{2,1}(t,\tau,x)*_{(x)}\big(|u(\tau,x)|^q-|\tilde u(\tau,x)|^q\big)d\tau\Big\|_{L^2(\mathbb{R}^n)}$$
$$\lesssim (1+B_2(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)}\|(u,v)-(\tilde u,\tilde v)\|_{X(t)}$$
$$\times\big(\|(u,v)\|^{q-1}_{X(t)}+\|(\tilde u,\tilde v)\|^{q-1}_{X(t)}\big),$$

$$\Big\|\partial_t\int_0^t E_{2,1}(t,\tau,x)*_{(x)}\big(|u(\tau,x)|^q-|\tilde u(\tau,x)|^q\big)d\tau\Big\|_{L^2(\mathbb{R}^n)}$$
$$\lesssim b_2(t)^{-1}(1+B_2(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-1}\|(u,v)-(\tilde u,\tilde v)\|_{X(t)}$$
$$\times\big(\|(u,v)\|^{q-1}_{X(t)}+\|(\tilde u,\tilde v)\|^{q-1}_{X(t)}\big),$$

$$\Big\||D|^{s_2-1}\partial_t\int_0^t E_{2,1}(t,\tau,x)*_{(x)}\big(|u(\tau,x)|^q-|\tilde u(\tau,x)|^q\big)d\tau\Big\|_{L^2(\mathbb{R}^n)}$$
$$\lesssim b_2(t)^{-1}(1+B_2(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s_2-1}{2}-1}\|(u,v)-(\tilde u,\tilde v)\|_{X(t)}$$
$$\times\big(\|(u,v)\|^{q-1}_{X(t)}+\|(\tilde u,\tilde v)\|^{q-1}_{X(t)}\big),$$

$$\Big\||D|^{s_2}\int_0^t E_{2,1}(t,\tau,x)*_{(x)}\big(|u(\tau,x)|^q-|\tilde u(\tau,x)|^q\big)d\tau\Big\|_{L^2(\mathbb{R}^n)}$$
$$\lesssim (1+B_2(t,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s_2}{2}}\|(u,v)-(\tilde u,\tilde v)\|_{X(t)}$$
$$\times\big(\|(u,v)\|^{q-1}_{X(t)}+\|(\tilde u,\tilde v)\|^{q-1}_{X(t)}\big),$$

where the conditions (3.51) and (3.52) are satisfied. These estimates lead to (3.54) which completes the proof. □

3.4. Large regular data

Theorem 3.10. *Let* $n\geq 4$, $(u_0,u_1),(v_0,v_1)\in\mathcal{A}_{m,s_1}\times\mathcal{A}_{m,s_2}$, $m\in[1,2)$, $s_2>s_1>\frac{n}{2}+1$, *and* $-1<r_1<r_2<1$. *Moreover, let*

$$p>s_1,\quad q>\tilde s_2,\quad \tilde q\geq\frac{2m}{n}\Big(\frac{s_2+1}{2}\Big)+1, \tag{3.66}$$

where

$$\tilde s_2\in(s_1,s_1+1),\quad \tilde s_2\leq s_2,\quad \tilde q=\tilde q_{r_1,r_2,m}=\frac{1+r_1}{1+r_2}\Big(q-\frac{m}{2}\Big)+\frac{m}{2}.$$

Then, there exists a constant ϵ_0 such that if

$$\|(u_0,u_1)\|_{\mathcal{A}_{m,s_1}}+\|(v_0,v_1)\|_{\mathcal{A}_{m,s_2}}\leq \epsilon_0,$$

then there exists a uniquely determined globally (in time) energy solution to (3.1) in

$$\begin{aligned}&\Big(\mathcal{C}\big([0,\infty),H^{s_1}(\mathbb{R}^n)\big)\cap\mathcal{C}^1\big([0,\infty),H^{s_1-1}(\mathbb{R}^n)\big)\Big)\\ &\qquad\times\Big(\mathcal{C}\big([0,\infty),H^{\tilde{s}_2}(\mathbb{R}^n)\big)\cap\mathcal{C}^1\big([0,t],H^{\tilde{s}_2-1}(\mathbb{R}^n)\big)\Big).\end{aligned}$$

Furthermore, the solution satisfies the following estimates:

$$\begin{aligned}
\|u(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \big(1+B_1(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)}\left(\|(u_0,u_1)\|_{\mathcal{A}_{m,s_1}}+\|(v_0,v_1)\|_{\mathcal{A}_{m,s_2}}\right),\\
\||D|^{s_1}u(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \big(1+B_1(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s_1}{2}}\\
&\qquad\times\left(\|(u_0,u_1)\|_{\mathcal{A}_{m,s_1}}+\|(v_0,v_1)\|_{\mathcal{A}_{m,s_2}}\right),\\
\|u_t(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim b_1(t)^{-1}\big(1+B_1(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-1}\\
&\qquad\times\left(\|(u_0,u_1)\|_{\mathcal{A}_{m,s_1}}+\|(v_0,v_1)\|_{\mathcal{A}_{m,s_2}}\right),\\
\||D|^{s_1-1}u_t(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim b_1(t)^{-1}\big(1+B_1(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-1-\frac{s_1-1}{2}}\\
&\qquad\left(\|(u_0,u_1)\|_{\mathcal{A}_{m,s_1}}+\|(v_0,v_1)\|_{\mathcal{A}_{m,s_2}}\right),
\end{aligned}$$

$$\begin{aligned}
\|v(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \big(1+B_2(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)}\left(\|(u_0,u_1)\|_{\mathcal{A}_{m,s_1}}+\|(v_0,v_1)\|_{\mathcal{A}_{m,s_2}}\right),\\
\||D|^{\tilde{s}_2}v(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \big(1+B_2(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{\tilde{s}_2}{2}}\\
&\qquad\times\left(\|(u_0,u_1)\|_{\mathcal{A}_{m,s_1}}+\|(v_0,v_1)\|_{\mathcal{A}_{m,s_2}}\right),\\
\|v_t(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim b_2(t)^{-1}\big(1+B_2(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-1}\\
&\qquad\times\left(\|(u_0,u_1)\|_{\mathcal{A}_{m,s_1}}+\|(v_0,v_1)\|_{\mathcal{A}_{m,s_2}}\right),\\
\||D|^{\tilde{s}-1}v_t(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim b_2(t)^{-1}\big(1+B_2(t,0)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-1-\frac{\tilde{s}_2-1}{2}}\\
&\qquad\left(\|(u_0,u_1)\|_{\mathcal{A}_{m,s_1}}+\|(v_0,v_1)\|_{\mathcal{A}_{m,s_2}}\right).
\end{aligned}$$

Remark 3.8. *Remark 3.7 remains valid for large regular data.*

Proof. To prove this theorem we choose the same space of solutions $X(t)$ was used in the proof of Theorem 3.9, by replacing $\tilde{s}_2$ instead of s_2, and following the same steps. The modifications appear in the conditions generated from the estimate of the following terms:

$$\||v(\tau,x)|^p\|_{\dot{H}^{s_1-1}(\mathbb{R}^n)},\ \||u(\tau,\cdot)|^q\|_{\dot{H}^{\tilde{s}_2-1}(\mathbb{R}^n)},$$

and

$$\||v(\tau,x)|^p-|\tilde{v}(\tau,x)|^p\|_{\dot{H}^{s_1-1}(\mathbb{R}^n)}\||u(\tau,x)|^q-|\tilde{u}(\tau,x)|^q\|_{\dot{H}^{\tilde{s}_2-1}(\mathbb{R}^n)}.$$

Then, as we did in the estimate (1.111), using fractional powers from Corollary A.3 and Proposition A.6 we get for $0\leq\tau\leq t$ the estimates:

$$\||v(\tau,x)|^p\|_{\dot{H}^{s_1-1}(\mathbb{R}^n)}\lesssim (1+B_2(\tau,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)p-\frac{s_1-1}{2}-\frac{s^*}{2}(p-1)}\|(u,v)\|^p_{X(t)},$$

$$\big\||u(\tau,\cdot)|^q\big\|_{\dot H^{\tilde s_2-1}(\mathbb{R}^n)} \lesssim (1+B_1(\tau,0))^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)p-\frac{\tilde s_2-1}{2}-\frac{s^*}{2}(q-1)}\|(u,v)\|^q_{X(t)},$$

$$\Big\||v(\tau,x)|^p-|\tilde v(\tau,x)|^p\Big\|_{\dot H^{s_1-1}(\mathbb{R}^n)} \lesssim (1+B_2(\tau,0))^{-\frac{n}{2m}p+\frac{n}{4}(p-1)-\frac{s_1-1}{2}-\frac{s^*}{2}(p-1)} \times\|(u,v)-(\tilde u,\tilde v)\|_{X(t)}\big(\|(u,v)\|^{p-1}_{X(t)}+\|(\tilde u,\tilde v)\|^{p-1}_{X(t)}\big),$$

$$\Big\||u(\tau,\cdot)|^q-|\tilde u(\tau,\cdot)|^q\Big\|_{\dot H^{\tilde s_2-1}(\mathbb{R}^n)} \lesssim (1+B_1(\tau,0))^{-\frac{n}{2m}p+\frac{n}{4}(q-1)-\frac{\tilde s_2-1}{2}-\frac{s^*}{2}(q-1)} \times\|(u,v)-(\tilde u,\tilde v)\|_{X(t)}\big(\|(u,v)\|^{q-1}_{X(t)}+\|(\tilde u,\tilde v)\|^{q-1}_{X(t)}\big),$$

where we used $p>s_1$ and $q>\tilde s_2$.
Taking into consideration these estimates, we get (3.57) to (3.64), for $\tilde s_2$ instead of s_2, under the following conditions:

$$-\frac{n}{2m}\tilde p+\frac{n}{4}-\Big(1-\frac{m}{2}\Big)\frac{r_2-r_1}{1+r_1}<-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s_1-1}{2}-1$$

which is satisfied by $p>s_1$ similarly to (3.56), and

$$-\frac{n}{2m}\tilde q+\frac{n}{4}<-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{s_1-1}{2}-1$$

which is equivalent to

$$\tilde q\geq\frac{2m}{n}\Big(\frac{s_2+1}{2}\Big)+1.$$

The last condition is supposed in the statement of the theorem. In this way we complete the proof. □

3.5. Concluding remarks

In this section we sketch possible generalizations of the results of this chapter. Let us choose the time-dependent coefficients $b_1=b_1(t)$ and $b_2=b_2(t)$ in such a way that the dissipation terms $b_1(t)u_t$ and $b_2(t)v_t$ become effective and the following condition is satisfied:

$$B_2(t,0)\approx B_1(t,0)^\alpha, \tag{3.67}$$

where α is a positive real number. It is clear that (3.67) covers a larger class of effective dissipation terms comparing with those are treated in the previous sections of this chapter.

Example 3.5. *The following coefficients satisfy the conditions of Hypothesis 1.1 and condition (3.67):*

1. $b_1(t)=\frac{\mu_1}{(1+t)^{r_1}}, b_2(t)=\frac{\mu_2}{(1+t)^{r_2}}$ *for some* $\mu_1,\mu_2>0$ *and* $r_1,r_2\in(-1,1)$, *where* $\alpha=\frac{1+r_2}{1+r_1}$,

2. $b_1(t) = \frac{\mu_1}{(1+t)^{r_1}}(\log(c_{r_1,\gamma_1}+t))^{\gamma_1}, b_2(t) = \frac{\mu_2}{(1+t)^{r_2}}(\log(c_{r_2,\gamma_2}+t))^{\gamma_2}$ *for some* $\mu_1, \mu_2 > 0$, $\gamma_1, \gamma_2 > 0$ *and* $\alpha = \frac{1+r_2}{1+r_1} = \frac{\gamma_2}{\gamma_1}$,

3. $b_1(t) = \frac{\mu_1}{(1+t)^{r_1}(\log(c_{r_1,\gamma_1}+t))^{\gamma_1}}, b_2(t) = \frac{\mu_2}{(1+t)^{r_2}(\log(c_{r_2,\gamma_2}+t))^{\gamma_2}}$ *for some* $\mu_1, \mu_1 > 0$, $\gamma_1, \gamma_1 > 0$ *and* $\alpha = \frac{1+r_2}{1+r_1} = \frac{\gamma_2}{\gamma_1}$,

where c_{r_1,γ_1} *and* c_{r_1,γ_1} *are sufficiently large positive constants.*

Using this new class of dissipation terms we summarize generalizations of the results of Sections 3.1 and 3.2 in the following tables:

Low regular data

$\tilde{p} < p_{Fuj,m}(n) < \tilde{q}$	$\tilde{q} < p_{Fuj,m}(n) < \tilde{p}$
Loss of decay in the estimate for u $\gamma_{n,m}(\tilde{p}) = -\frac{n}{2m}(\tilde{p}-1)+1$ interaction condition $\frac{n}{2} > m\Big(\frac{\tilde{q}+\frac{1}{\alpha}}{\tilde{p}\tilde{q}-1+(\tilde{p}-1)\frac{1-\alpha}{\alpha}}\Big)$	Loss of decay in the estimate for v $\gamma_{n,m}(\tilde{q}) = -\frac{n}{2m}(\tilde{q}-1)+1$ interaction condition $\frac{n}{2} > m\Big(\frac{\tilde{p}+\alpha}{\tilde{p}\tilde{q}-1+(\tilde{q}-1)(\alpha-1)}\Big)$

Data from the energy space

$s=1$	$\tilde{p} < p_{Fuj,m}(n) < \tilde{q}$	$\tilde{q} < p_{Fuj,m}(n) < \tilde{p}$
$\alpha < 1$ $\tilde{q} = \tilde{q}_m$ $\tilde{p} = \tilde{p}$	Loss of decay in the estimate for u $\gamma_{n,m}(\tilde{p}) = -\frac{n}{2m}(\tilde{p}-1)+1$ interaction condition $\frac{n}{2} > m\Big(\frac{\tilde{q}+1+\frac{m}{2}\left(\frac{\alpha-1}{\alpha}\right)}{\tilde{p}\tilde{q}-1+\frac{m}{2}(\tilde{p}-1)\left(\frac{\alpha-1}{\alpha}\right)}\Big)$	Loss of decay in the estimate for v $\gamma_{n,m}(\tilde{q}) = -\frac{n}{2m}(\tilde{q}-1)+1+\varepsilon$ interaction condition $\frac{n}{2} > m\Big(\frac{\tilde{p}+\alpha}{\tilde{p}\tilde{q}-\alpha+\tilde{q}(\alpha-1)}\Big)$
$\alpha > 1$ $\tilde{q} = \tilde{q}$ $\tilde{p} = \tilde{p}_m$	Loss of decay in the estimate for u $\gamma_{n,m}(\tilde{p}) = -\frac{n}{2m}(\tilde{p}-1)+1+\varepsilon$ interaction condition $\frac{n}{2} > m\Big(\frac{\tilde{q}+\frac{1}{\alpha}}{\tilde{p}\tilde{q}-\frac{1}{\alpha}+\tilde{p}\left(\frac{\alpha-1}{\alpha}\right)}\Big)$	Loss of decay in the estimate for v $\gamma_{n,m}(\tilde{q}) = -\frac{n}{2m}(\tilde{q}-1)+1$ interaction condition $\frac{n}{2} > m\Big(\frac{\tilde{p}+1+\frac{m}{2}(\alpha-1)}{\tilde{p}\tilde{q}-1+\frac{m}{2}(\tilde{q}-1)(\alpha-1)}\Big)$

where

$$\begin{aligned}\tilde{q}_m &= \frac{1}{\alpha}\left(q-\frac{m}{2}\right)+\frac{m}{2}, \quad \tilde{q}_{m=2} = \tilde{q} = \frac{1}{\alpha}(q-1)+1,\\ \tilde{p}_m &= \alpha\left(p-\frac{m}{2}\right)+\frac{m}{2}, \quad \tilde{p}_{m=2} = \tilde{p} = \alpha(p-1)+1.\end{aligned} \tag{3.68}$$

We can present possible generalizations of the results of Sections 3.3 and 3.4, too, by using the modified exponents of power nonlinearities (3.68) and dissipation terms satisfying condition (3.67).

4. Weakly coupled systems of semilinear damped waves with different scale-invariant time-dependent dissipation terms

In this chapter we are interested in the following weakly coupled system

$$\begin{aligned} &u_{tt} - \Delta u + \tfrac{\mu_1}{1+t}u_t = |v|^p, \qquad v_{tt} - \Delta v + \tfrac{\mu_2}{1+t}v_t = |u|^q, \\ &u(0,x) = u_0(x),\ u_t(0,x) = u_1(x),\ v(0,x) = v_0(x),\ v_t(0,x) = v_1(x), \end{aligned} \tag{4.1}$$

where $(t,x) \in [0,\infty) \times \mathbb{R}^n$ and $\mu_1, \mu_2 > 1$ are real constants. Before studying this system (4.1), let us recall some previous results for the corresponding wave models with scale-invariant time-dependent dissipation term. Firstly, we consider the following homogeneous Cauchy problem

$$u_{tt} - \Delta u + \frac{\mu}{1+t}u_t = 0, \qquad u(0,x) = u_0(x),\ u_t(0,x) = u_1(x). \tag{4.2}$$

In [68], $L^p(\mathbb{R}^n) - L^q(\mathbb{R}^n)$ estimates for solutions were proved for every positive μ. Many papers are concerned with the following semilinear Cauchy problem

$$u_{tt} - \Delta u + \frac{\mu}{1+t}u_t = |u|^p, \qquad u(0,x) = u_0(x),\ u_t(0,x) = u_1(x). \tag{4.3}$$

In [6, 47] and [52] the authors showed that the situation depends strongly on the value of μ, in other words, the transition of μ from 1 to ∞ describes the change from a hyperbolic to a parabolic like model from the point of decay estimates for solutions. Furthermore, they proved that the decay rate of solutions to (4.2) for large μ is the same that is obtained for solutions of the classical damped wave equation (0.5). A particular case of the Cauchy problem (4.3) with $\mu = 2$ was treated in [8].

Recently, in [54] the estimates for the solution to the Cauchy problem (4.3) were proved for different classes of regularity of the data are introduced in the previous chapters: low regular data, data from energy space, data from Sobolev spaces with suitable regularity and large regular data. We summarize these results in the following theorems.

Theorem 4.1. *Let us assume the data $(u_0,u_1) \in \mathcal{A}_{m,s}$ with $s > 0$. Then the solution u to the Cauchy problem (4.2) satisfies for $\mu > 1$ the following decay estimates:*

For $s \geq 0$:

$$
\begin{aligned}
\||D|^s u(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \|(u_0,u_1)\|_{\mathcal{A}_{m,s}} \\
&\times \begin{cases} (1+t)^{-\frac{2-m}{2m}n-s} & \text{if} \quad \mu > \frac{2-m}{m}n+2s, \\ (1+t)^{-\frac{\mu}{2}}\left(1+\log(1+t)\right)^{\frac{2-m}{2m}} & \text{if} \quad \mu = \frac{2-m}{m}n+2s, \\ (1+t)^{-\frac{\mu}{2}} & \text{if} \quad \mu < \frac{2-m}{m}n+2s; \end{cases}
\end{aligned}
\tag{4.4}
$$

and for $s \geq 1$:

$$
\begin{aligned}
\||D|^{s-1} u_t(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \|(u_0,u_1)\|_{\mathcal{A}_{m,s}} \\
&\times \begin{cases} (1+t)^{-\frac{2-m}{2m}n-s} & \text{if} \quad \mu > \frac{2-m}{m}n+2s, \\ (1+t)^{-\frac{\mu}{2}}\left(1+\log(1+t)\right)^{\frac{2-m}{2m}} & \text{if} \quad \mu = \frac{2-m}{m}n+2s, \\ (1+t)^{-\frac{\mu}{2}} & \text{if} \quad \mu < \frac{2-m}{m}n+2s. \end{cases}
\end{aligned}
\tag{4.5}
$$

In order to use Duhamel's principle in the next sections, we consider the family of parameter-dependent Cauchy problems

$$
v_{tt} - \Delta v + \frac{\mu}{1+t} v_t = 0, \quad v(\tau,x) = 0, \;\; v_t(\tau,x) = v_1(\tau,x), \quad 0 \leq \tau \leq t. \tag{4.6}
$$

Theorem 4.2. *Let us assume $v_1 \in H^{\max\{s-1;0\}}(\mathbb{R}^n) \cap L^m(\mathbb{R}^n)$ with $s > 0$. Then the solution v to the Cauchy problem (4.6) satisfies for for $\mu > 1$ the following decay estimates for $s \geq 0$:*

$$
\begin{aligned}
\||D|^s v(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \left(\|v_1\|_{L^m(\mathbb{R}^n)} + (1+\tau)^{\frac{2-m}{2m}n+[s-1]^+}\|v_1\|_{\dot{H}^{[s-1]^+}(\mathbb{R}^n)}\right)(1+\tau) \\
&\times \begin{cases} (1+t)^{-\frac{2-m}{2m}n-s} & \text{if} \quad \mu > \frac{2-m}{m}n+2s, \\ (1+t)^{-\frac{\mu}{2}}\left(1+\log(1+t)\right)^{\frac{2-m}{2m}} & \text{if} \quad \mu = \frac{2-m}{m}n+2s, \\ (1+t)^{-\frac{\mu}{2}}(1+\tau)^{-\frac{2-m}{2m}n+\frac{\mu}{2}-s} & \text{if} \quad \mu < \frac{2-m}{m}n+2s; \end{cases}
\end{aligned}
\tag{4.7}
$$

and for $s \geq 1$:

$$
\begin{aligned}
&\||D|^{s-1} v_t(t,\cdot)\|_{L^2(\mathbb{R}^n)} \\
&\lesssim \Big(\|v_1\|_{L^m(\mathbb{R}^n)} + (1+\tau)^{\frac{2-m}{2m}n+s-1}\|v_1\|_{\dot{H}^{s-1}(\mathbb{R}^n)} + (1+\tau)^{\frac{2-m}{2m}n+[s-2]^+}\|v_1\|_{\dot{H}^{[s-2]^+}(\mathbb{R}^n)}\Big) \\
&\qquad \times(1+\tau) \begin{cases} (1+t)^{-\frac{2-m}{2m}n-s} & \text{if} \quad \mu > \frac{2-m}{m}n+2s, \\ (1+t)^{-\frac{\mu}{2}}\left(1+\log(1+t)\right)^{\frac{2-m}{2m}} & \text{if} \quad \mu = \frac{2-m}{m}n+2s, \\ (1+t)^{-\frac{\mu}{2}}(1+\tau)^{-\frac{2-m}{2m}n+\frac{\mu}{2}-s} & \text{if} \quad \mu < \frac{2-m}{m}n+2s. \end{cases}
\end{aligned}
\tag{4.8}
$$

4.1. Low regular data

In this section we are interested in the Cauchy problem (4.1), where the initial data are supposed to have low regularity, or in other words, the data belongs to the Sobolev space $H^s(\mathbb{R}^n)$ for $s \in (0,1)$, with additional regularity $L^m(\mathbb{R}^n)$ for $m \in [1,2)$.

From the estimates of Theorem 4.1 and further considerations, we remark the existence of five cases corresponding to the value of μ. These cases are the followings:

$$\mu > \frac{2-m}{m}n + 2s, \quad \mu = \frac{2-m}{m}n + 2s, \quad \frac{2-m}{m}n + 2s > \mu > \frac{2-m}{m}n,$$
$$\mu = \frac{2-m}{m}n \text{ and } \mu < \frac{2-m}{m}n.$$

These cases generate for the system (4.1) a lot of cases corresponding to the values of μ_1 and μ_2. In Theorem 4.3 we restrict ourselves to three cases which from our point of view are more interesting and important. The remaining cases will be treated in Remark 4.1.

Lemma 4.1. *Let p satisfy the conditions*

$$\begin{array}{lll} \frac{2}{m} \le p & \text{if} \;\; n = 1 \;\; \text{and} & s \in [\frac{1}{2}, 1), \\ \frac{2}{m} \le p \le p_{GN,s}(n) & \text{otherwise.} & \end{array} \tag{4.9}$$

Then, the following statements are valid:
If

$$M(\tau,u) = (1+\tau)^{\frac{2-m}{2m}n}\|u(\tau,\cdot)\|_{L^2(\mathbb{R}^n)} + (1+\tau)^{\frac{2-m}{2m}n+s}\||D|^s u(\tau,\cdot)\|_{L^2(\mathbb{R}^n)},$$

then

$$\||u(\tau,x)|^p\|_{L^m(\mathbb{R}^n)} \lesssim (1+\tau)^{-\frac{n}{m}(p-1)}M(\tau,u)^p,$$
$$\||u(\tau,x)|^p\|_{L^2(\mathbb{R}^n)} \lesssim (1+\tau)^{-\frac{n}{m}p+\frac{n}{2}}M(\tau,u)^p.$$

These estimates imply

$$\||u(\tau,x)|^p\|_{L^m(\mathbb{R}^n)} + (1+\tau)^{\frac{2-m}{2m}n}\||u(\tau,x)|^p\|_{L^2(\mathbb{R}^n)} \lesssim (1+\tau)^{-\frac{n}{m}(p-1)}M(\tau,u)^p. \tag{4.10}$$

If

$$M(\tau,u) = (1+\tau)^{\frac{\mu}{2}}\left(\|u(\tau,\cdot)\|_{L^2(\mathbb{R}^n)} + \||D|^s u(\tau,\cdot)\|_{L^2(\mathbb{R}^n)}\right),$$

then

$$\||u(\tau,x)|^p\|_{L^m(\mathbb{R}^n)} + \||u(\tau,x)|^p\|_{L^2(\mathbb{R}^n)} \lesssim (1+\tau)^{-\frac{\mu}{2}p}M(\tau,u)^p.$$

These estimates imply

$$\begin{aligned} \||u(\tau,x)|^p\|_{L^m(\mathbb{R}^n)} &+ (1+\tau)^{\frac{2-m}{2m}n}\||u(\tau,x)|^p\|_{L^2(\mathbb{R}^n)} \\ &\lesssim (1+\tau)^{-\frac{\mu}{2}p+\frac{2-m}{2m}n}M(\tau,u)^p. \end{aligned} \tag{4.11}$$

Proof. To prove Lemma 4.1, we follow the same steps are used to prove estimates (1.34) and (1.35) by using the Gagliardo-Nirenberg inequality. □

Theorem 4.3. *Let $n \le \frac{4s}{2-m}$. The data $(u_0,u_1),(v_0,v_1)$ are supposed to belong to $\mathcal{A}_{m,s} \times \mathcal{A}_{m,s}$, $s \in (0,1)$, $m \in [1,2)$ and $\min\{\mu_1;\mu_2\} > 1$. Moreover, let the exponents p and q of the power nonlinearities satisfy the condition (4.9) and*

$$\min\{p;q\} > p_{Fuj,m}(n) \quad \text{if} \quad \min\{\mu_1;\mu_2\} > \frac{2-m}{m}n + 2s, \tag{4.12}$$

$$p > \frac{4}{\mu_2} + \frac{2-m}{\mu_2 m}n, q > \frac{2m}{n} + \frac{m}{2} + \frac{n\mu_2}{2m} \quad \textit{if} \quad \mu_1 > \frac{2-m}{m}n + 2s, \mu_2 < \frac{2-m}{m}n, \tag{4.13}$$

$$p > \frac{\mu_1}{\mu_2} + \frac{4}{\mu_2}, q > \frac{\mu_2}{\mu_1} + \frac{4}{\mu_1} \quad \textit{if} \quad \max\{\mu_1; \mu_2\} < \frac{2-m}{m}n. \tag{4.14}$$

Then, there exists a uniquely determined globally (in time) Sobolev solution to (4.1) in

$$\left(\mathcal{C}([0,\infty), H^s(\mathbb{R}^n))\right)^2.$$

Furthermore, the solution satisfies for $s \in [0,1)$ the following decay estimates:

$$\begin{aligned} \||D|^s u(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \big(\|(u_0,u_1)\|_{\mathcal{A}_{m,s}} + \|(v_0,v_1)\|_{\mathcal{A}_{m,s}}\big) \\ &\times \begin{cases} (1+t)^{-\frac{2-m}{2m}n-s} & \textit{if} \quad \mu_1 > \frac{2-m}{m}n + 2s, \\ (1+t)^{-\frac{\mu_1}{2}} & \textit{if} \quad \mu_1 < \frac{2-m}{m}n; \end{cases} \end{aligned} \tag{4.15}$$

and

$$\begin{aligned} \||D|^s v(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \big(\|(u_0,u_1)\|_{\mathcal{A}_{m,s}} + \|(v_0,v_1)\|_{\mathcal{A}_{m,s}}\big) \\ &\times \begin{cases} (1+t)^{-\frac{2-m}{2m}n-s} & \textit{if} \quad \mu_2 > \frac{2-m}{m}n + 2s, \\ (1+t)^{-\frac{\mu_2}{2}} & \textit{if} \quad \mu_2 < \frac{2-m}{m}n. \end{cases} \end{aligned} \tag{4.16}$$

Example 4.1. *If we consider the model (4.1) for $n = 2$, $m = \frac{11}{10}$ and $s = \frac{9}{10}$, then by using (4.12), (4.13) and (4.14) from Theorem 4.3 we obtain the following statements:*

$$\begin{aligned} &\textit{if} \quad \mu_1 = 4, \mu_2 = 3, \quad \textit{then} \quad \min\{p; q\} > p_{Fuj,\frac{11}{10}}(2) = \tfrac{21}{10}, \\ &\textit{if} \quad \mu_1 = 4, \mu_2 = 2, \quad \textit{then} \quad p > \tfrac{53}{22} \sim 2.4, q > \tfrac{763}{220} \sim 3.46, \\ &\textit{if} \quad \mu_1 = 2, \mu_2 = \frac{3}{2}, \quad \textit{then} \quad p > 4, q > \tfrac{11}{4} = 2.75. \end{aligned}$$

Proof. Let us define the space of solutions $X(t)$ as follows:

$$X(t) = \left\{(u,v) \in \left(\mathcal{C}([0,t], H^s(\mathbb{R}^n))\right)^2\right\}$$

with the norm

$$\|(u,v)\|_{X(t)} = \sup_{\tau\in[0,t]} \big\{M_1(\tau,u) + M_2(\tau,v)\big\},$$

where $M_1(\tau,u)$ and $M_2(\tau,u)$ will be defined in the treatment of each case. Let N be the mapping on $X(t)$ defined by (3.5). Then our aim is to prove the following estimates:

$$\|N(u,v)\|_{X(t)} \lesssim \|(u_0,u_1)\|_{\mathcal{A}_{m,s}} + \|(v_0,v_1)\|_{\mathcal{A}_{m,s}} + \|(u,v)\|^p_{X(t)} + \|(u,v)\|^q_{X(t)}, \tag{4.17}$$

$$\begin{aligned} \|N(u,v) - N(\tilde{u},\tilde{v})\|_{X(t)} &\lesssim \|(u,v) - (\tilde{u},\tilde{v})\|_{X(t)} \\ &\times \big(\|(u,v)\|^{p-1}_{X(t)} + \|(\tilde{u},\tilde{v})\|^{p-1}_{X(t)} + \|(u,v)\|^{q-1}_{X(t)} + \|(\tilde{u},\tilde{v})\|^{q-1}_{X(t)}\big). \end{aligned} \tag{4.18}$$

From the definition of the norm of the solution space $X(t)$, which we will define in each theorem in correspondence with the main goals, we can immediately obtain

$$\|(u^{ln}, v^{ln})\|_{X(t)} \lesssim \|(u_0,u_1)\|_{\mathcal{A}_{m,s}} + \|(v_0,v_1)\|_{\mathcal{A}_{m,s}}.$$

We complete the proof of all results separately by showing the inequality

$$\|(u^{nl}, v^{nl})\|_{X(t)} \lesssim \|(u,v)\|^p_{X(t)} + \|(u,v)\|^q_{X(t)}. \tag{4.19}$$

To prove (4.18), we follow the same steps from the proofs of the theorems of Section 3.1 of Chapter 3. Here the regularity parameter of the data is not more than 1. For this reason, there are no additional restrictions to the conditions generated from the proof of the existence part.

1. $\underline{\min\{\mu_1; \mu_2\} > \frac{2-m}{m}n + 2s}$:
   We choose

$$M_1(\tau,u) = (1+\tau)^{\frac{2-m}{2m}n}\|u(\tau,\cdot)\|_{L^2(\mathbb{R}^n)} + (1+\tau)^{\frac{2-m}{2m}n+s}\||D|^s u(\tau,\cdot)\|_{L^2(\mathbb{R}^n)},$$

$$M_2(\tau,v) = (1+\tau)^{\frac{2-m}{2m}n}\|v(\tau,\cdot)\|_{L^2(\mathbb{R}^n)} + (1+\tau)^{\frac{2-m}{2m}n+s}\||D|^s v(\tau,\cdot)\|_{L^2(\mathbb{R}^n)}.$$

   For the first component u^{nl} we use the estimate (4.7) for $\mu_1 > \frac{2-m}{m}n + 2s$ and the estimate (4.10) to obtain

$$\begin{aligned}
&\||D|^s u^{nl}(t,\cdot)\|_{L^2(\mathbb{R}^n)} \\
&\lesssim \int_0^t \big(\||v(\tau,x)|^p\|_{L^m(\mathbb{R}^n)} + (1+\tau)^{\frac{2-m}{2m}n}\||v(\tau,x)|^p\|_{L^2(\mathbb{R}^n)}\big)(1+\tau)(1+t)^{-\frac{2-m}{2m}n-s}d\tau \\
&\lesssim (1+t)^{-\frac{2-m}{2m}n-s}\int_0^t \big(\||v(\tau,x)|^p\|_{L^m(\mathbb{R}^n)} + (1+\tau)^{\frac{2-m}{2m}n}\||v(\tau,x)|^p\|_{L^2(\mathbb{R}^n)}\big)(1+\tau)d\tau \\
&\lesssim \|(u,v)\|^p_{X(t)}(1+t)^{-\frac{2-m}{2m}n-s}\int_0^t (1+\tau)^{-\frac{n}{m}(p-1)+1}d\tau \\
&\lesssim \|(u,v)\|^p_{X(t)}(1+t)^{-\frac{2-m}{2m}n-s},
\end{aligned}$$

   where we used $p > p_{Fuj,m}(n)$. Then we get

$$\||D|^s u^{nl}(t,\cdot)\|_{L^2(\mathbb{R}^n)} \lesssim \|(u,v)\|^p_{X(t)}(1+t)^{-\frac{2-m}{2m}n-s}. \tag{4.20}$$

   In the same way we prove

$$\|u^{nl}(t,\cdot)\|_{L^2(\mathbb{R}^n)} \lesssim \|(u,v)\|^p_{X(t)}(1+t)^{-\frac{2-m}{2m}n}. \tag{4.21}$$

   For v^{nl} we use the estimate (4.7) for $\mu_2 > \frac{2-m}{m}n + 2s$ and the estimate (4.10) to obtain

$$\begin{aligned}
&\||D|^s v^{nl}(t,\cdot)\|_{L^2(\mathbb{R}^n)} \\
&\lesssim \int_0^t \big(\||u(\tau,\cdot)|^q\|_{L^m(\mathbb{R}^n)} + (1+\tau)^{\frac{2-m}{2m}n}\||u(\tau,\cdot)|^q\|_{L^2(\mathbb{R}^n)}\big)(1+\tau)(1+t)^{-\frac{2-m}{2m}n-s}d\tau \\
&\lesssim (1+t)^{-\frac{2-m}{2m}n-s}\int_0^t \big(\||u(\tau,\cdot)|^q\|_{L^m(\mathbb{R}^n)} + (1+\tau)^{\frac{2-m}{2m}n}\||u(\tau,\cdot)|^q\|_{L^2(\mathbb{R}^n)}\big)(1+\tau)d\tau \\
&\lesssim \|(u,v)\|^q_{X(t)}(1+t)^{-\frac{2-m}{2m}n-s}\int_0^t (1+\tau)^{-\frac{n}{m}(q-1)+1}d\tau \\
&\lesssim \|(u,v)\|^q_{X(t)}(1+t)^{-\frac{2-m}{2m}n-s},
\end{aligned}$$

where we used $q > p_{Fuj,m}(n)$. Then we get

$$\||D|^s v^{nl}(t,\cdot)\|_{L^2(\mathbb{R}^n)} \lesssim \|(u,v)\|^q_{X(t)}(1+t)^{-\frac{2-m}{2m}n-s}. \tag{4.22}$$

In the same way we prove

$$\|v^{nl}(t,\cdot)\|_{L^2(\mathbb{R}^n)} \lesssim \|(u,v)\|^q_{X(t)}(1+t)^{-\frac{2-m}{2m}n}. \tag{4.23}$$

From (4.20) to (4.23) we complete the proof of (4.19).

2. $\underline{\mu_1 > \frac{2-m}{m}n + 2s, \ \ \mu_2 < \frac{2-m}{m}n}$:
   We choose

$$M_1(\tau,u) = (1+\tau)^{\frac{2-m}{2m}n}\|u(\tau,\cdot)\|_{L^2(\mathbb{R}^n)} + (1+\tau)^{\frac{2-m}{2m}n+s}\||D|^s u(\tau,\cdot)\|_{L^2(\mathbb{R}^n)},$$

$$M_2(\tau,v) = (1+\tau)^{\frac{\mu_2}{2}}\big(\|v(\tau,\cdot)\|_{L^2(\mathbb{R}^n)} + \||D|^s v(\tau,\cdot)\|_{L^2(\mathbb{R}^n)}\big).$$

For the first component u^{nl} we use the estimate (4.7) for $\mu_1 > \frac{2-m}{m}n + 2s$ and the estimate (4.11) to obtain

$$\begin{aligned}
&\||D|^s u^{nl}(t,\cdot)\|_{L^2(\mathbb{R}^n)}\\
&\lesssim \int_0^t \big(\||v(\tau,x)|^p\|_{L^m(\mathbb{R}^n)} + (1+\tau)^{\frac{2-m}{2m}n}\||v(\tau,x)|^p\|_{L^2(\mathbb{R}^n)}\big)(1+\tau)(1+t)^{-\frac{2-m}{2m}n-s}d\tau\\
&\lesssim (1+t)^{-\frac{2-m}{2m}n-s}\int_0^t \big(\||v(\tau,x)|^p\|_{L^m(\mathbb{R}^n)} + (1+\tau)^{\frac{2-m}{2m}n}\||v(\tau,x)|^p\|_{L^2(\mathbb{R}^n)}\big)(1+\tau)d\tau\\
&\lesssim \|(u,v)\|^p_{X(t)}(1+t)^{-\frac{2-m}{2m}n-s}\int_0^t (1+\tau)^{-\frac{\mu_2}{2}p+\frac{2-m}{2m}n+1}d\tau\\
&\lesssim \|(u,v)\|^p_{X(t)}(1+t)^{-\frac{2-m}{2m}n-s},
\end{aligned}$$

where we used (4.13) for the exponent p. Then we get

$$\||D|^s u^{nl}(t,\cdot)\|_{L^2(\mathbb{R}^n)} \lesssim \|(u,v)\|^p_{X(t)}(1+t)^{-\frac{2-m}{2m}n-s}. \tag{4.24}$$

In the same way we prove

$$\|u^{nl}(t,\cdot)\|_{L^2(\mathbb{R}^n)} \lesssim \|(u,v)\|^p_{X(t)}(1+t)^{-\frac{2-m}{2m}n}. \tag{4.25}$$

For v^{nl} we use the estimate (4.7) for $\mu_2 < \frac{2-m}{m}n$ and the estimate (4.10) to obtain

$$\begin{aligned}
&\||D|^s v^{nl}(t,\cdot)\|_{L^2(\mathbb{R}^n)}\\
&\lesssim \int_0^t \big(\||u(\tau,\cdot)|^q\|_{L^m(\mathbb{R}^n)} + (1+\tau)^{\frac{2-m}{2m}n}\||u(\tau,\cdot)|^q\|_{L^2(\mathbb{R}^n)}\big)\\
&\qquad\qquad \times(1+t)^{-\frac{\mu_2}{2}}(1+\tau)^{-\frac{2-m}{2m}n+\frac{\mu_2}{2}-s+1}d\tau\\
&\lesssim \|(u,v)\|^q_{X(t)}(1+t)^{-\frac{\mu_2}{2}}\int_0^t (1+\tau)^{-\frac{n}{m}(q-1)-\frac{2-m}{2m}n+\frac{\mu_2}{2}-s+1}d\tau\\
&\lesssim \|(u,v)\|^q_{X(t)}(1+t)^{-\frac{\mu_2}{2}},
\end{aligned}$$

where we used (4.13) for the exponent q. Then we get

$$\||D|^s v^{nl}(t,\cdot)\|_{L^2(\mathbb{R}^n)} \lesssim \|(u,v)\|^q_{X(t)}(1+t)^{-\frac{\mu_2}{2}}. \tag{4.26}$$

In the same way we prove

$$\|v^{nl}(t,\cdot)\|_{L^2(\mathbb{R}^n)} \lesssim \|(u,v)\|^q_{X(t)}(1+t)^{-\frac{\mu_2}{2}}. \tag{4.27}$$

From (4.24) to (4.27) we complete the proof of (4.19).

3. $\max\{\mu_1;\mu_2\} < \frac{2-m}{m}n$:
   We choose

$$M_1(\tau,u) = (1+\tau)^{\frac{\mu_1}{2}}\big(\|u(\tau,\cdot)\|_{L^2(\mathbb{R}^n)} + \||D|^s u(\tau,\cdot)\|_{L^2(\mathbb{R}^n)}\big),$$

$$M_2(\tau,v) = (1+\tau)^{\frac{\mu_2}{2}}\big(\|v(\tau,\cdot)\|_{L^2(\mathbb{R}^n)} + \||D|^s v(\tau,\cdot)\|_{L^2(\mathbb{R}^n)}\big).$$

For u^{nl} we use the estimate (4.7) for $\mu_1 < \frac{2-m}{m}n$ and the estimate (4.11) to obtain

$$\begin{aligned}
&\||D|^s u^{nl}(t,\cdot)\|_{L^2(\mathbb{R}^n)}\\
&\lesssim \int_0^t \big(\||v(\tau,x)|^p\|_{L^m(\mathbb{R}^n)} + (1+\tau)^{\frac{2-m}{2m}n}\||v(\tau,x)|^p\|_{L^2(\mathbb{R}^n)}\big)\\
&\qquad\qquad \times(1+t)^{-\frac{\mu_1}{2}}(1+\tau)^{-\frac{2-m}{2m}n+\frac{\mu_1}{2}-s+1}d\tau\\
&\lesssim \|(u,v)\|^p_{X(t)}(1+t)^{-\frac{\mu_1}{2}}\int_0^t (1+\tau)^{-\frac{2-m}{2m}n+\frac{\mu_1}{2}-s+1-\frac{\mu_2}{2}p+\frac{2-m}{2m}n}d\tau\\
&\lesssim \|(u,v)\|^p_{X(t)}(1+t)^{-\frac{\mu_1}{2}},
\end{aligned}$$

where we used $p > \frac{4-2s}{\mu_2} + \frac{\mu_1}{\mu_2}$ which is included in (4.14). Then, we get

$$\||D|^s u^{nl}(t,\cdot)\|_{L^2(\mathbb{R}^n)} \lesssim \|(u,v)\|^p_{X(t)}(1+t)^{-\frac{\mu_1}{2}}. \tag{4.28}$$

In the same way, for $s=0$ we can prove

$$\|u^{nl}(t,\cdot)\|_{L^2(\mathbb{R}^n)} \lesssim \|(u,v)\|^p_{X(t)}(1+t)^{-\frac{\mu_1}{2}}, \tag{4.29}$$

where we use the condition (4.14).
Analogously, using (4.14) for q we can prove

$$\||D|^s v^{nl}(t,\cdot)\|_{L^2(\mathbb{R}^n)} \lesssim \|(u,v)\|^q_{X(t)}(1+t)^{-\frac{\mu_2}{2}}, \tag{4.30}$$

$$\|v^{nl}(t,\cdot)\|_{L^2(\mathbb{R}^n)} \lesssim \|(u,v)\|^q_{X(t)}(1+t)^{-\frac{\mu_2}{2}}. \tag{4.31}$$

From (4.28) to (4.31) we complete the proof of (4.19).

The proof of the theorem is complete. □

Remark 4.1. *In the following table we will show all conditions generated by the different cases for the values of μ_1 and μ_2.*

case	Admissible range for p	Admissible range for q
1- $\mu_1 > \frac{2-m}{m}n + 2s$, $\mu_2 = \frac{2-m}{m}n + 2s$	$p > p_{Fuj,m}(n)$	$q > p_{Fuj,m}(n)$
2- $\mu_1 > \frac{2-m}{m}n + 2s$, $\frac{2-m}{m}n < \mu_2 < \frac{2-m}{m}n + 2s$	$p > \min\big\{\frac{4m}{(2-m)n} + 1; \big(2+\frac{n}{m}\big)\big(\frac{2}{\mu_2-2s+n}\big)\big\}$	$q > p_{Fuj,m}(n)$
3- $\mu_1 > \frac{2-m}{m}n + 2s$, $\mu_2 = \frac{2-m}{m}n$	$p > 1 + \frac{4}{\mu_2}$	$q > p_{Fuj,m}(n)$
4- $\mu_1 = \mu_2 = \frac{2-m}{m}n + 2s$	$p > p_{Fuj,m}(n)$	$q > p_{Fuj,m}(n)$
5- $\mu_1 = \frac{2-m}{m}n + 2s$, $\frac{2-m}{m}n < \mu_2 < \frac{2-m}{m}n + 2s$	$p > \min\big\{\frac{4m}{(2-m)n} + 1; \big(2+\frac{n}{m}\big)\big(\frac{2}{\mu_2-2s+n}\big)\big\}$	$q > p_{Fuj,m}(n)$
6- $\mu_1 = \frac{2-m}{m}n + 2s$, $\mu_2 = \frac{2-m}{m}n$	$p > 1 + \frac{4}{\mu_2}$	$q > p_{Fuj,m}(n)$
7- $\mu_1 = \frac{2-m}{m}n + 2s$, $\mu_2 < \frac{2-m}{m}n$	$p > \frac{4}{\mu_2} + \frac{2-m}{\mu_2 m}n$	$q > \frac{2m}{n} + \frac{m}{2} + \frac{n\mu_2}{2m}$
8- $\frac{2-m}{m}n < \mu_1 < \frac{2-m}{m}n + 2s$, $\frac{2-m}{m}n < \mu_2 < \frac{2-m}{m}n + 2s$	$p > \min\big\{\frac{4m}{(2-m)n} + 1; \big(2+\frac{n}{m}\big)\big(\frac{2}{\mu_2-2s+n}\big)\big\}$	$q > \min\big\{\frac{4m}{(2-m)n} + 1; \big(2+\frac{n}{m}\big)\big(\frac{2}{\mu_1-2s+n}\big)\big\}$
9- $\frac{2-m}{m}n < \mu_1 < \frac{2-m}{m}n + 2s$, $\mu_2 = \frac{2-m}{m}n$	$p > 1 + \frac{4}{\mu_2}$	$q > \min\big\{\frac{4m}{(2-m)n} + 1; \big(2+\frac{n}{m}\big)\big(\frac{2}{\mu_1-2s+n}\big)\big\}$
10- $\frac{2-m}{m}n < \mu_1 < \frac{2-m}{m}n + 2s$, $\mu_2 < \frac{2-m}{m}n$	$p > \frac{4}{\mu_2} + \frac{2-m}{m\mu_2}n$	$q > \min\big\{\frac{2m}{(2-m)n}\big(2-\frac{\mu_2}{2}\big); \frac{4+n+\mu_1}{\mu_1-2s+n}\big\}$
11- $\mu_1 = \mu_2 = \frac{2-m}{m}n$	$p > 1 + \frac{4}{\mu_2}$	$q > 1 + \frac{4}{\mu_1}$
12- $\mu_1 = \frac{2-m}{m}n$, $\mu_2 < \frac{2-m}{m}n$	$p > 1 + \frac{4}{\mu_2}$	$q > \frac{\mu_2}{\mu_1} + \frac{4}{\mu_1}$

4.2. Data from energy space

If the data are in the energy space, then we get for $s = 1$ a similar case to the case of the previous section because the estimates for $\||D|^{s=1}u(t,\cdot)\|_{L^2(\mathbb{R}^n)}$ and $\|u_t(t,\cdot)\|_{L^2(\mathbb{R}^n)}$ coincide with those of the previous section. Moreover, we obtain the global existence in time of energy solutions. Consequently, we have the following result.

Theorem 4.4. *Let $n \le \frac{4}{2-m}$, $(u_0,u_1),(v_0,v_1) \in \mathcal{A}_{m,1}\times\mathcal{A}_{m,1}$, $m \in [1,2)$ and $\min\{\mu_1;\mu_2\} > 1$. Moreover, let the exponents p and q of power nonlinearities satisfy the condition (4.9) and*

$$\begin{array}{ll}
\min\{p;q\} > p_{Fuj,m}(n) & \textit{if } \min\{\mu_1;\mu_2\} > \frac{2-m}{m}n+2,\\
p > \frac{4}{\mu_2} + \frac{2-m}{\mu_2 m}n,\ q > \frac{2m}{n} + \frac{m}{2} + \frac{n\mu_2}{2m} & \textit{if } \mu_1 > \frac{2-m}{m}n+2,\ \mu_2 < \frac{2-m}{m}n,\\
p > \frac{\mu_1}{\mu_2} + \frac{4}{\mu_2},\ q > \frac{\mu_2}{\mu_1} + \frac{4}{\mu_1} & \textit{if } \max\{\mu_1;\mu_2\} < \frac{2-m}{m}n.
\end{array}$$

Then, there exists a small constant ϵ_0 such that if

$$\|(u_0,u_1)\|_{\mathcal{A}_{m,1}} + \|(v_0,v_1)\|_{\mathcal{A}_{m,1}} \le \epsilon_0,$$

then there exists a uniquely determined globally (in time) energy solution to (4.1) in

$$\big(\mathcal{C}\big([0,\infty),H^1(\mathbb{R}^n)\big)\cap\mathcal{C}^1\big([0,\infty),L^2(\mathbb{R}^n)\big)\big)^2.$$

Furthermore, the solution satisfies the following decay estimates:

$$\begin{aligned}\|u_t(t,\cdot)\|_{L^2(\mathbb{R}^n)}+\|\nabla u(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \big(\|(u_0,u_1)\|_{\mathcal{A}_{m,1}}+\|(v_0,v_1)\|_{\mathcal{A}_{m,1}}\big)\\ \times\begin{cases}(1+t)^{-\frac{2-m}{2m}n-1} & \text{if}\quad \mu_1>\frac{2-m}{m}n+2,\\ (1+t)^{-\frac{\mu_1}{2}} & \text{if}\quad \mu_1<\frac{2-m}{m}n,\end{cases}\end{aligned}$$

$$\begin{aligned}\|u(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \big(\|(u_0,u_1)\|_{\mathcal{A}_{m,1}}+\|(v_0,v_1)\|_{\mathcal{A}_{m,1}}\big)\\ &\qquad\times\begin{cases}(1+t)^{-\frac{2-m}{2m}n} & \text{if}\quad \mu_1>\frac{2-m}{m}n+2,\\ (1+t)^{-\frac{\mu_1}{2}} & \text{if}\quad \mu_1<\frac{2-m}{m}n,\end{cases}\end{aligned}$$

and

$$\begin{aligned}\|v_t(t,\cdot)\|_{L^2(\mathbb{R}^n)}+\|\nabla v(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \big(\|(u_0,u_1)\|_{\mathcal{A}_{m,1}}+\|(v_0,v_1)\|_{\mathcal{A}_{m,1}}\big)\\ \times\begin{cases}(1+t)^{-\frac{2-m}{2m}n-1} & \text{if}\quad \mu_2>\frac{2-m}{m}n+2,\\ (1+t)^{-\frac{\mu_2}{2}} & \text{if}\quad \mu_2<\frac{2-m}{m}n,\end{cases}\end{aligned}$$

$$\begin{aligned}\|v(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \big(\|(u_0,u_1)\|_{\mathcal{A}_{m,1}}+\|(v_0,v_1)\|_{\mathcal{A}_{m,1}}\big)\\ &\qquad\times\begin{cases}(1+t)^{-\frac{2-m}{2m}n} & \text{if}\quad \mu_2>\frac{2-m}{m}n+2,\\ (1+t)^{-\frac{\mu_2}{2}} & \text{if}\quad \mu_2<\frac{2-m}{m}n.\end{cases}\end{aligned}$$

Example 4.2. *If we consider the system (4.1) for $n=2$ and $m=2$, then for given $\mu_1=\mu_2=\frac{21}{10}$ we get after using the last case the following admissible ranges for p and q:*

$$\min\{p;q\}>3.$$

4.3. Data from Sobolev spaces with suitable regularity

This section is devoted to the case where the data are from Sobolev spaces with suitable regularity. We will treat the same cases of the previous sections corresponding to the values of μ_1 and μ_2. In the following lemma we will provide some estimates which are important tools in the proofs of our main results.

Lemma 4.2. *Let $p>\lceil s\rceil$ satisfy the following condition:*

$$\begin{array}{lcll} 2< & p & <\infty & \text{if}\quad s\in[\frac{n}{2},\frac{n}{2}+1],\\ 2< & p & \leq 1+\frac{2}{n-2s} & \text{if}\quad s\in(2,\frac{n}{2}).\end{array}\tag{4.32}$$

Then, the following statements are valid:
If

$$\begin{aligned}M(\tau,u) &= (1+\tau)^{\frac{2-m}{2m}n}\|u(\tau,\cdot)\|_{L^2(\mathbb{R}^n)}+(1+\tau)^{\frac{2-m}{2m}n+1}\|u_t(\tau,\cdot)\|_{L^2(\mathbb{R}^n)}\\ &\quad+(1+\tau)^{\frac{2-m}{2m}n+s}\big(\||D|^s u(\tau,\cdot)\|_{L^2(\mathbb{R}^n)}+\||D|^{s-1}u_t(\tau,\cdot)\|_{L^2(\mathbb{R}^n)}\big),\end{aligned}$$

then

$$\begin{aligned}
&\||u(\tau,x)|^p\|_{L^m(\mathbb{R}^n)} \lesssim (1+\tau)^{-\frac{n}{m}(p-1)}M(\tau,u)^p,\\
&\||u(\tau,x)|^p\|_{L^2(\mathbb{R}^n)} \lesssim (1+\tau)^{-\frac{n}{m}p+\frac{n}{2}}M(\tau,u)^p,\\
&\||u(\tau,x)|^p\|_{\dot{H}^{s-1}(\mathbb{R}^n)} \lesssim (1+\tau)^{-\frac{n}{m}p+\frac{n}{2}-(s-1)}M(\tau,u)^p,
\end{aligned}$$

$$\begin{aligned}
\big\||u(\tau,x)|^p-|\tilde{u}(\tau,x)|^p\big\|_{\dot{H}^{s-1}(\mathbb{R}^n)} &\lesssim (1+B(\tau,0))^{-\frac{n}{m}p+\frac{n}{2}-(s-1)}\\
&\quad\times M(\tau,v-\tilde{v})\big(M(\tau,u)^{p-1}+M(\tau,\tilde{u})^{p-1}\big).
\end{aligned} \tag{4.33}$$

These estimates imply

$$\begin{aligned}
&\Big(\||u|^p\|_{L^m(\mathbb{R}^n)}+(1+\tau)^{\frac{2-m}{2m}n+s-1}\||u|^p\|_{\dot{H}^{s-1}(\mathbb{R}^n)}+(1+\tau)^{\frac{2-m}{2m}n+s-2}\||u|^p\|_{\dot{H}^{s-2}(\mathbb{R}^n)}\Big)\\
&\qquad\times(1+\tau)\lesssim(1+\tau)^{-\frac{n}{m}(p-1)+1}M(\tau,u)^p.
\end{aligned} \tag{4.34}$$

If

$$\begin{aligned}
M(\tau,u) \;=\; &(1+\tau)^{\frac{\mu}{2}}\left(\|u(\tau,\cdot)\|_{L^2(\mathbb{R}^n)}+\|u_t(\tau,\cdot)\|_{L^2(\mathbb{R}^n)}\right.\\
&\left.+\||D|^s u(\tau,\cdot)\|_{L^2(\mathbb{R}^n)}+\||D|^{s-1}u_t(\tau,\cdot)\|_{L^2(\mathbb{R}^n)}\right),
\end{aligned}$$

then

$$\||u(\tau,x)|^p\|_{L^m(\mathbb{R}^n)}+\||u(\tau,x)|^p\|_{L^2(\mathbb{R}^n)}+\||u(\tau,x)|^p\|_{\dot{H}^s(\mathbb{R}^n)}\lesssim(1+\tau)^{-\frac{\mu}{2}p}M(\tau,u)^p,$$

$$\begin{aligned}
&\big\||u(\tau,x)|^p-|\tilde{u}(\tau,x)|^p\big\|_{\dot{H}^{s-1}(\mathbb{R}^n)}\\
&\quad\lesssim(1+B(\tau,0))^{-\frac{\mu}{2}p}M(\tau,v-\tilde{v})\big(M(\tau,u)^{p-1}+M(\tau,\tilde{u})^{p-1}\big).
\end{aligned} \tag{4.35}$$

These estimates imply

$$\begin{aligned}
&\Big(\||u|^p\|_{L^m(\mathbb{R}^n)}+(1+\tau)^{\frac{2-m}{2m}n+s-1}\||u|^p\|_{\dot{H}^{s-1}(\mathbb{R}^n)}+(1+\tau)^{\frac{2-m}{2m}n+s-2}\||u|^p\|_{\dot{H}^{s-2}(\mathbb{R}^n)}\Big)\\
&\qquad\times(1+\tau)\lesssim(1+\tau)^{-\frac{\mu}{2}p+\frac{2-m}{2m}n+s}M(\tau,u)^p.
\end{aligned} \tag{4.36}$$

Proof. Following the same steps to prove (1.62) and (1.78) and by using the fractional chain rule one can prove the desired statements. □

Theorem 4.5. *Let $n\geq 4$. The regularity parameters s_1 and s_2 satisfy the following conditions:*

$$s_1,s_2\in\big(2,\frac{n}{2}+1\big],\;\; 0<s_2-s_1<1 \;\;\textit{and}\;\; \lceil s_1\rceil\neq\lceil s_2\rceil.$$

The data (u_0,u_1) and (v_0,v_1) are supposed to belong to $\mathcal{A}_{m,s_1}\times\mathcal{A}_{m,s_2}$ with $m\in[1,2)$. Furthermore, we assume for the exponents p and q the following conditions:

$$\begin{array}{llll}
\lceil s_1\rceil<p, & \lceil s_2\rceil<q & \textit{if} & n\leq 2s_1,\\
\lceil s_1\rceil<p, & \lceil s_2\rceil<q\leq 1+\frac{2}{n-2s_1} & \textit{if} & 2s_1<n\leq 2s_2,\\
\lceil s_1\rceil<p\leq 1+\frac{2}{n-2s_2}, & \lceil s_2\rceil<q\leq 1+\frac{2}{n-2s_1} & \textit{if} & n>2s_2.
\end{array} \tag{4.37}$$

and

$$\min\{p;q\}>p_{Fuj,m}(n) \quad \textit{if} \quad \mu_1>\frac{2-m}{m}n+2s_1,\;\mu_2>\frac{2-m}{m}n+2s_2,$$

$$p > \frac{2+2s_2}{\mu_2} + \frac{2-m}{\mu_2 m}n,\ q > \frac{2m}{n} + \frac{m}{2} + \frac{n\mu_2}{2m} \quad \textit{if} \quad \mu_1 > \frac{2-m}{m}n + 2s_1,\ \mu_2 < \frac{2-m}{m}n,$$

$$p > \frac{\mu_1}{\mu_2} + \frac{2+2s_2}{\mu_2},\ q > \frac{\mu_2}{\mu_1} + \frac{2+2s_1}{\mu_1} \quad \textit{if} \quad \max\{\mu_1;\mu_2\} < \frac{2-m}{m}n.$$

Then, there exists a uniquely determined globally (in time) energy solution to (4.1) in

$$\begin{aligned}&\big(\mathcal{C}\big([0,\infty),H^{s_1}(\mathbb{R}^n)\big)\cap\mathcal{C}^1\big([0,\infty),H^{s_1-1}(\mathbb{R}^n)\big)\big)\\ &\quad\times\big(\mathcal{C}\big([0,\infty),H^{s_2}(\mathbb{R}^n)\big)\cap\mathcal{C}^1\big([0,\infty),H^{s_2-1}(\mathbb{R}^n)\big)\big).\end{aligned}$$

Furthermore, the solution satisfies the following decay estimates:

$$\begin{aligned}\|u(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \big(\|(u_0,u_1)\|_{\mathcal{A}_{m,s_1}} + \|(v_0,v_1)\|_{\mathcal{A}_{m,s_2}}\big)\\ &\quad\times\begin{cases}(1+t)^{-\frac{2-m}{2m}n} & \textit{if} \ \ \mu_1 > \frac{2-m}{m}n + 2s_1,\\ (1+t)^{-\frac{\mu_1}{2}} & \textit{if} \ \ \mu_1 < \frac{2-m}{m}n,\end{cases}\end{aligned}$$

$$\begin{aligned}\|u_t(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \big(\|(u_0,u_1)\|_{\mathcal{A}_{m,s_1}} + \|(v_0,v_1)\|_{\mathcal{A}_{m,s_2}}\big)\\ &\quad\times\begin{cases}(1+t)^{-\frac{2-m}{2m}n-1} & \textit{if} \ \ \mu_1 > \frac{2-m}{m}n + 2s_1,\\ (1+t)^{-\frac{\mu_1}{2}} & \textit{if} \ \ \mu_1 < \frac{2-m}{m}n,\end{cases}\end{aligned}$$

$$\begin{aligned}&\||D|^{s_1}u(\tau,\cdot)\|_{L^2(\mathbb{R}^n)} + \||D|^{s_1-1}u_t(\tau,\cdot)\|_{L^2(\mathbb{R}^n)} \lesssim \big(\|(u_0,u_1)\|_{\mathcal{A}_{m,s_1}} + \|(v_0,v_1)\|_{\mathcal{A}_{m,s_2}}\big)\\ &\qquad\times\begin{cases}(1+t)^{-\frac{2-m}{2m}n-s_1} & \textit{if} \ \ \mu_1 > \frac{2-m}{m}n + 2s_1,\\ (1+t)^{-\frac{\mu_1}{2}} & \textit{if} \ \ \mu_1 < \frac{2-m}{m}n,\end{cases}\end{aligned}$$

$$\begin{aligned}\|v(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \big(\|(u_0,u_1)\|_{\mathcal{A}_{m,s_1}} + \|(v_0,v_1)\|_{\mathcal{A}_{m,s_2}}\big)\\ &\quad\times\begin{cases}(1+t)^{-\frac{2-m}{2m}n} & \textit{if} \ \ \mu_2 > \frac{2-m}{m}n + 2s_2,\\ (1+t)^{-\frac{\mu_2}{2}} & \textit{if} \ \ \mu_2 < \frac{2-m}{m}n,\end{cases}\end{aligned}$$

$$\begin{aligned}\|v_t(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \big(\|(u_0,u_1)\|_{\mathcal{A}_{m,s_1}} + \|(v_0,v_1)\|_{\mathcal{A}_{m,s_2}}\big)\\ &\quad\times\begin{cases}(1+t)^{-\frac{2-m}{2m}n-1} & \textit{if} \ \ \mu_2 > \frac{2-m}{m}n + 2s_2,\\ (1+t)^{-\frac{\mu_2}{2}} & \textit{if} \ \ \mu_2 < \frac{2-m}{m}n,\end{cases}\end{aligned}$$

$$\begin{aligned}&\||D|^{s_2}v(\tau,\cdot)\|_{L^2(\mathbb{R}^n)} + \||D|^{s_2-1}v_t(\tau,\cdot)\|_{L^2(\mathbb{R}^n)} \lesssim \big(\|(u_0,u_1)\|_{\mathcal{A}_{m,s_1}} + \|(v_0,v_1)\|_{\mathcal{A}_{m,s_2}}\big)\\ &\qquad\times\begin{cases}(1+t)^{-\frac{2-m}{2m}n-s_2} & \textit{if} \ \ \mu_2 > \frac{2-m}{m}n + 2s_2,\\ (1+t)^{-\frac{\mu_2}{2}} & \textit{if} \ \ \mu_2 < \frac{2-m}{m}n.\end{cases}\end{aligned}$$

Proof. To prove this theorem we follow the same steps of the proof of Theorem 4.3. Then our main goal is to prove (4.19) which implies (4.17). To prove (4.18) we use the estimates (4.33) or (4.35) and follow the same steps used to prove Theorem 2.15, in particular, inequality (2.59). We split the proof into three cases.

1. $\mu_1 > \frac{2-m}{m}n + 2s_1,\ \mu_2 > \frac{2-m}{m}n + 2s_2$:
   We choose

$$\begin{aligned} M_1(\tau,u) &= (1+\tau)^{\frac{2-m}{2m}n}\|u(\tau,\cdot)\|_{L^2(\mathbb{R}^n)} + (1+\tau)^{\frac{2-m}{2m}n+1}\|u_t(\tau,\cdot)\|_{L^2(\mathbb{R}^n)} \\ &\quad +(1+\tau)^{\frac{2-m}{2m}n+s_1}\left(\||D|^{s_1}u(\tau,\cdot)\|_{L^2(\mathbb{R}^n)} + \||D|^{s_1-1}u_t(\tau,\cdot)\|_{L^2(\mathbb{R}^n)}\right), \end{aligned}$$

$$\begin{aligned} M_2(\tau,v) &= (1+\tau)^{\frac{2-m}{2m}n}\|v(\tau,\cdot)\|_{L^2(\mathbb{R}^n)} + (1+\tau)^{\frac{2-m}{2m}n+1}\|v_t(\tau,\cdot)\|_{L^2(\mathbb{R}^n)} \\ &\quad +(1+\tau)^{\frac{2-m}{2m}n+s_2}\left(\||D|^{s_2}v(\tau,\cdot)\|_{L^2(\mathbb{R}^n)} + \||D|^{s_2-1}v_t(\tau,\cdot)\|_{L^2(\mathbb{R}^n)}\right). \end{aligned}$$

For the first component u^{nl}, we estimate the most complicate norm (from our point of view) which is $\||D|^{s_1-1}u_t^{nl}(\tau,\cdot)\|_{L^2(\mathbb{R}^n)}$. After using the estimate (4.8) for $\mu_1 > \frac{2-m}{m}n + 2s_1$ and the estimate (4.34) we obtain

$$\begin{aligned} &\||D|^{s_1-1}u_t^{nl}(t,\cdot)\|_{L^2(\mathbb{R}^n)} \\ &\lesssim \int_0^t \Big(\||v(\tau,x)|^p\|_{L^m(\mathbb{R}^n)} + (1+\tau)^{\frac{2-m}{2m}n+s_1-1}\||v(\tau,x)|^p\|_{\dot{H}^{s_1-1}(\mathbb{R}^n)} \\ &\qquad +(1+\tau)^{\frac{2-m}{2m}n+s_1-2}\|||v(\tau,x)|^p\|_{\dot{H}^{s_1-2}(\mathbb{R}^n)}\Big)(1+\tau)(1+t)^{-\frac{2-m}{2m}n-s_1}d\tau \\ &\lesssim \|(u,v)\|_{X(t)}^p(1+t)^{-\frac{2-m}{2m}n-s_1}\int_0^t (1+\tau)^{-\frac{n}{m}(p-1)+1}d\tau \\ &\lesssim \|(u,v)\|_{X(t)}^p(1+t)^{-\frac{2-m}{2m}n-s_1}, \end{aligned}$$

where we used the condition $p > p_{Fuj,m}(n)$. Then we get

$$\||D|^{s_1-1}u_t^{nl}(t,\cdot)\|_{L^2(\mathbb{R}^n)} \lesssim \|(u,v)\|_{X(t)}^p(1+t)^{-\frac{2-m}{2m}n-s_1}. \tag{4.38}$$

In the same way we get

$$\|u^{nl}(t,\cdot)\|_{L^2(\mathbb{R}^n)} \lesssim \|(u,v)\|_{X(t)}^p(1+t)^{-\frac{2-m}{2m}n}, \tag{4.39}$$

$$\|u_t^{nl}(t,\cdot)\|_{L^2(\mathbb{R}^n)} \lesssim \|(u,v)\|_{X(t)}^p(1+t)^{-\frac{2-m}{2m}n-1}, \tag{4.40}$$

$$\||D|^{s_1}u^{nl}(t,\cdot)\|_{L^2(\mathbb{R}^n)} \lesssim \|(u,v)\|_{X(t)}^p(1+t)^{-\frac{2-m}{2m}n-s_1}. \tag{4.41}$$

Analogously, for the second component v^{nl}, we use $q > p_{Fuj,m}(n)$ to derive the estimates

$$\||D|^{s_2-1}v_t^{nl}(t,\cdot)\|_{L^2(\mathbb{R}^n)} \lesssim \|(u,v)\|_{X(t)}^q(1+t)^{-\frac{2-m}{2m}n-s_2}, \tag{4.42}$$

$$\|v^{nl}(t,\cdot)\|_{L^2(\mathbb{R}^n)} \lesssim \|(u,v)\|_{X(t)}^q(1+t)^{-\frac{2-m}{2m}n}, \tag{4.43}$$

$$\|v_t^{nl}(t,\cdot)\|_{L^2(\mathbb{R}^n)} \lesssim \|(u,v)\|_{X(t)}^q(1+t)^{-\frac{2-m}{2m}n-1}, \tag{4.44}$$

$$\||D|^{s_2}v^{nl}(t,\cdot)\|_{L^2(\mathbb{R}^n)} \lesssim \|(u,v)\|_{X(t)}^q(1+t)^{-\frac{2-m}{2m}n-s_2}. \tag{4.45}$$

From (4.38) to (4.45) we get (4.19).

2. $\underline{\mu_1 > \frac{2-m}{m}n + 2s,\ \mu_2 < \frac{2-m}{m}n}$:
We choose

$$\begin{aligned} M_1(\tau,u) &= (1+\tau)^{\frac{2-m}{2m}n}\|u(\tau,\cdot)\|_{L^2(\mathbb{R}^n)} + (1+\tau)^{\frac{2-m}{2m}n+1}\|u_t(\tau,\cdot)\|_{L^2(\mathbb{R}^n)} \\ &\quad + (1+\tau)^{\frac{2-m}{2m}n+s_1}\left(\||D|^{s_1}u(\tau,\cdot)\|_{L^2(\mathbb{R}^n)} + \||D|^{s_1-1}u_t(\tau,\cdot)\|_{L^2(\mathbb{R}^n)}\right), \end{aligned}$$

$$\begin{aligned} M_2(\tau,v) &= (1+\tau)^{\frac{\mu_2}{2}}\left(\|v(\tau,\cdot)\|_{L^2(\mathbb{R}^n)} + \|v_t(\tau,\cdot)\|_{L^2(\mathbb{R}^n)}\right. \\ &\quad \left. + \||D|^{s_2}v(\tau,\cdot)\|_{L^2(\mathbb{R}^n)} + \||D|^{s_2-1}v_t(\tau,\cdot)\|_{L^2(\mathbb{R}^n)}\right). \end{aligned}$$

We begin to estimate the norm $\||D|^{s_1-1}u_t^{nl}(\tau,\cdot)\|_{L^2(\mathbb{R}^n)}$. After using the estimate (4.8) for $\mu_1 > \frac{2-m}{m}n + 2s_1$ and the estimate (4.36) we have

$$\begin{aligned} &\||D|^{s_1-1}u_t^{nl}(t,\cdot)\|_{L^2(\mathbb{R}^n)} \\ &\lesssim \int_0^t \Big(\||v(\tau,x)|^p\|_{L^m(\mathbb{R}^n)} + (1+\tau)^{\frac{2-m}{2m}n+s_1-1}\||v(\tau,x)|^p\|_{\dot H^{s_1-1}(\mathbb{R}^n)} \\ &\qquad + (1+\tau)^{\frac{2-m}{2m}n+s_1-2}\|\||v(\tau,x)|^p\|_{\dot H^{s_1-2}(\mathbb{R}^n)}\Big)(1+\tau)(1+t)^{-\frac{2-m}{2m}n-s_1}d\tau \\ &\lesssim \|(u,v)\|_{X(t)}^p(1+t)^{-\frac{2-m}{2m}n-s_1}\int_0^t (1+\tau)^{-\frac{\mu_2}{2}p+\frac{2-m}{2m}n+s_1}d\tau \\ &\lesssim \|(u,v)\|_{X(t)}^p(1+t)^{-\frac{2-m}{2m}n-s_1}, \end{aligned}$$

where we used the condition $p > \frac{2+2s_2}{\mu_2} + \frac{2-m}{\mu_2 m}n$. Then we get (4.38) and in similar way one can prove (4.39) to (4.41).
to estimate the norm $\||D|^{s_2-1}v_t^{nl}(\tau,\cdot)\|_{L^2(\mathbb{R}^n)}$ we use the estimate (4.8) for $\mu_2 < \frac{2-m}{m}n$ and the estimate (4.34). Then it follows

$$\begin{aligned} &\||D|^{s_2-1}v_t^{nl}(t,\cdot)\|_{L^2(\mathbb{R}^n)} \\ &\lesssim \int_0^t \Big(\||u(\tau,\cdot)|^q\|_{L^m(\mathbb{R}^n)} + (1+\tau)^{\frac{2-m}{2m}n+s_2-1}\||u(\tau,\cdot)|^q\|_{\dot H^{s_2-1}(\mathbb{R}^n)} \\ &\qquad + (1+\tau)^{\frac{2-m}{2m}n+s_2-2}\|\||u(\tau,\cdot)|^q\|_{\dot H^{s_2-2}(\mathbb{R}^n)}\Big)(1+\tau)(1+t)^{-\frac{2-m}{2m}n+\frac{\mu_2}{2}-s_2}d\tau \\ &\lesssim \|(u,v)\|_{X(t)}^q(1+t)^{-\frac{\mu_2}{2}}\int_0^t (1+\tau)^{-\frac{n}{m}(q-1)-\frac{2-m}{2m}n+\frac{\mu_2}{2}-s_2}d\tau \\ &\lesssim \|(u,v)\|_{X(t)}^q(1+t)^{-\frac{\mu_2}{2}}, \end{aligned}$$

where we used the condition $q > \frac{(2-2s_2)m}{n} + \frac{m}{2} + \frac{n\mu_2}{2m}$. We may conclude

$$\||D|^{s_2-1}v_t^{nl}(t,\cdot)\|_{L^2(\mathbb{R}^n)} \lesssim \|(u,v)\|_{X(t)}^q(1+t)^{-\frac{\mu_2}{2}}. \tag{4.46}$$

Analogously, by using the condition $q > \frac{2m}{n} + \frac{m}{2} + \frac{n\mu_2}{2m}$, we can prove

$$\|v^{nl}(t,\cdot)\|_{L^2(\mathbb{R}^n)} + \|v_t^{nl}(t,\cdot)\|_{L^2(\mathbb{R}^n)} + \||D|^{s_2}v^{nl}(t,\cdot)\|_{L^2(\mathbb{R}^n)} \lesssim \|(u,v)\|_{X(t)}^q(1+t)^{-\frac{\mu_2}{2}}. \tag{4.47}$$

In this way we complete the proof of the second case.

3. $\underline{\max\{\mu_1;\mu_2\} < \frac{2-m}{m}n}$:
   We choose

$$\begin{aligned} M_1(\tau,u) &= (1+\tau)^{\frac{\mu_1}{2}} \big(\|u(\tau,\cdot)\|_{L^2(\mathbb{R}^n)} + \|u_t(\tau,\cdot)\|_{L^2(\mathbb{R}^n)} \\ &\qquad + \||D|^{s_1}u(\tau,\cdot)\|_{L^2(\mathbb{R}^n)} + \||D|^{s_1-1}u_t(\tau,\cdot)\|_{L^2(\mathbb{R}^n)} \big), \end{aligned}$$

$$\begin{aligned} M_2(\tau,v) &= (1+\tau)^{\frac{\mu_2}{2}} \big(\|v(\tau,\cdot)\|_{L^2(\mathbb{R}^n)} + \|v_t(\tau,\cdot)\|_{L^2(\mathbb{R}^n)} \\ &\qquad + \||D|^{s_2}v(\tau,\cdot)\|_{L^2(\mathbb{R}^n)} + \||D|^{s_2-1}v_t(\tau,\cdot)\|_{L^2(\mathbb{R}^n)} \big). \end{aligned}$$

   For u^{nl} we use the estimate (4.8) for $\mu_1 < \frac{2-m}{m}n$ and the estimate (4.36) to obtain

$$\begin{aligned} &\||D|^{s_1-1}u_t^{nl}(t,\cdot)\|_{L^2(\mathbb{R}^n)} \\ &\lesssim \int_0^t \Big(\||v(\tau,x)|^p\|_{L^m(\mathbb{R}^n)} + (1+\tau)^{\frac{2-m}{2m}n+s_1-1}\||v(\tau,x)|^p\|_{\dot H^{s_1-1}(\mathbb{R}^n)} \\ &\qquad + (1+\tau)^{\frac{2-m}{2m}n+s_1-2}\|\||v(\tau,x)|^p\|_{\dot H^{s_1-2}(\mathbb{R}^n)} \Big) (1+\tau)^{-\frac{2-m}{2m}n+\frac{\mu_1}{2}-s_1+1} d\tau \\ &\lesssim \|(u,v)\|_{X(t)}^p (1+t)^{-\frac{\mu_1}{2}} \int_0^t (1+\tau)^{-\frac{2-m}{2m}n+\frac{\mu_1}{2}-s_1-\frac{\mu}{2}p+\frac{2-m}{2m}n+s_2} d\tau \\ &\lesssim \|(u,v)\|_{X(t)}^q (1+t)^{-\frac{\mu_1}{2}}, \end{aligned}$$

   where we used the condition $p > \frac{\mu_1}{\mu_2} + \frac{2+2s_2-2s_1}{\mu_2}$. Then,

$$\||D|^{s_1-1}u_t^{nl}(t,\cdot)\|_{L^2(\mathbb{R}^n)} \lesssim \|(u,v)\|_{X(t)}^p (1+t)^{-\frac{\mu_1}{2}}. \tag{4.48}$$

   In the same way we can prove for $p > \frac{\mu_1}{\mu_2} + \frac{2+2s_2}{\mu_2}$ the estimate

$$\|u^{nl}(t,\cdot)\|_{L^2(\mathbb{R}^n)} + \|u_t^{nl}(t,\cdot)\|_{L^2(\mathbb{R}^n)} + \||D|^{s_1}u^{nl}(t,\cdot)\|_{L^2(\mathbb{R}^n)} \lesssim \|(u,v)\|_{X(t)}^p (1+t)^{-\frac{\mu_1}{2}}. \tag{4.49}$$

   Analogously, under the condition $q > \frac{\mu_2}{\mu_1} + \frac{2+2s_1}{\mu_1}$ it follows

$$\begin{aligned} &\||D|^{s_2-1}v_t(t,\cdot)\|_{L^2(\mathbb{R}^n)} + \|v(t,\cdot)\|_{L^2(\mathbb{R}^n)} + \|v_t(t,\cdot)\|_{L^2(\mathbb{R}^n)} \\ &\qquad + \||D|^{s_2}v_t(t,\cdot)\|_{L^2(\mathbb{R}^n)} \lesssim \|(u,v)\|_{X(t)}^q (1+t)^{-\frac{\mu_2}{2}}. \end{aligned} \tag{4.50}$$

   From (4.48) to (4.50) we get (4.19).

This completes the proof. □

4.4. Large regular data

Comparing this section, where the data are supposed to have large regularity, with the previous section, we feel differences in the treatment only if μ_1 and μ_2 are sufficiently large. For this reason we restrict ourselves to formulate the results without giving a proof. The proof is very similar to the proof of Theorem 4.5.

Lemma 4.3. *Let $p > s$. Then using the rule for fractional powers from Proposition A.6 in the Appendix the following estimates hold: If*

$$\begin{aligned} M(\tau,u) &= (1+\tau)^{\frac{2-m}{2m}n}\|u(\tau,\cdot)\|_{L^2(\mathbb{R}^n)} + (1+\tau)^{\frac{2-m}{2m}n+1}\|u_t(\tau,\cdot)\|_{L^2(\mathbb{R}^n)} \\ &\quad +(1+\tau)^{\frac{2-m}{2m}n+s}\left(\||D|^s u(\tau,\cdot)\|_{L^2(\mathbb{R}^n)} + \||D|^{s-1}u_t(\tau,\cdot)\|_{L^2(\mathbb{R}^n)}\right), \end{aligned}$$

then

$$\begin{aligned} &\||u(\tau,x)|^p\|_{L^m(\mathbb{R}^n)} \lesssim (1+\tau)^{-\frac{n}{m}(p-1)}M(\tau,u)^p, \\ &\||u(\tau,x)|^p\|_{L^2(\mathbb{R}^n)} \lesssim (1+\tau)^{-\frac{n}{m}p+\frac{m}{2}}M(\tau,u)^p, \\ &\||u(\tau,x)|^p\|_{\dot H^s(\mathbb{R}^n)} \lesssim (1+\tau)^{-\frac{2-m}{2m}np-s-s^*(p-1)}M(\tau,u)^p, \end{aligned}$$

$$\begin{aligned} \big\||u(\tau,x)|^p - |\tilde u(\tau,x)|^p\big\|_{\dot H^{s-1}(\mathbb{R}^n)} &\lesssim (1+B(\tau,0))^{-\frac{2-m}{2m}np-(s-1)-s^*(p-1)} \\ &\quad \times M(\tau,v-\tilde v)\big(M(\tau,u)^{p-1}+M(\tau,\tilde u)^{p-1}\big), \end{aligned}$$

where we assume $s^* < \frac{n}{2}$.
These estimates imply with the inequality (1.115) the estimate

$$\begin{aligned} &\Big(\||u|^p\|_{L^m(\mathbb{R}^n)} + (1+\tau)^{\frac{2-m}{2m}n+s-1}\||u|^p\|_{\dot H^{s-1}(\mathbb{R}^n)} + (1+\tau)^{\frac{2-m}{2m}n+s-2}\||u|^p\|_{\dot H^{s-2}(\mathbb{R}^n)}\Big) \\ &\quad \times(1+\tau) \lesssim (1+\tau)^{-\frac{n}{m}(p-1)+s}M(\tau,u)^p. \end{aligned}$$

If

$$\begin{aligned} M(\tau,u) &= (1+\tau)^{\frac{\mu}{2}}\left(\|u(\tau,\cdot)\|_{L^2(\mathbb{R}^n)} + \|u_t(\tau,\cdot)\|_{L^2(\mathbb{R}^n)}\right. \\ &\quad \left.+\||D|^s u(\tau,\cdot)\|_{L^2(\mathbb{R}^n)} + \||D|^{s-1}u_t(\tau,\cdot)\|_{L^2(\mathbb{R}^n)}\right), \end{aligned}$$

then

$$\||u(\tau,x)|^p\|_{L^m(\mathbb{R}^n)} + \||u(\tau,x)|^p\|_{L^2(\mathbb{R}^n)} + \||u(\tau,x)|^p\|_{\dot H^s(\mathbb{R}^n)} \lesssim (1+\tau)^{-\frac{\mu}{2}p}M(\tau,u)^p,$$

$$\begin{aligned} \big\||u(\tau,x)|^p - |\tilde u(\tau,x)|^p\big\|_{\dot H^{s-1}(\mathbb{R}^n)} &\lesssim (1+B(\tau,0))^{-\frac{\mu}{2}p} \\ &\quad \times M(\tau,v-\tilde v)\big(M(\tau,u)^{p-1}+M(\tau,\tilde u)^{p-1}\big). \end{aligned}$$

These estimates imply the estimate

$$\begin{aligned} &\Big(\||u|^p\|_{L^m(\mathbb{R}^n)} + (1+\tau)^{\frac{2-m}{2m}n+s-1}\||u|^p\|_{\dot H^{s-1}(\mathbb{R}^n)} + (1+\tau)^{\frac{2-m}{2m}n+s-2}\||u|^p\|_{\dot H^{s-2}(\mathbb{R}^n)}\Big) \\ &\quad \times(1+\tau) \lesssim (1+\tau)^{-\frac{\mu}{2}p+\frac{2-m}{2m}n+s}M(\tau,u)^p. \end{aligned}$$

Theorem 4.6. *Let $n \geq 4$. The data (u_0,u_1) and (v_0,v_1) are supposed to belong to $\mathcal{A}_{m,s_1}\times\mathcal{A}_{m,s_2}$ with $m\in[1,2)$ and $s_2 > s_1 > \frac{n}{2}+1$. Moreover, we assume*

$$p > s_1 \quad \text{and} \quad q > \tilde s_2, \tag{4.51}$$

where $\tilde s_2 \in [s_1, s_1+1]$ and $\tilde s_2 \leq s_2$. Furthermore, we assume for the exponents p and q the following conditions:

$$p > 1+\frac{(\tilde s_2+1)m}{n},\ q > 1+\frac{(s_1+1)m}{n} \quad \text{if} \quad \mu_1 > \frac{2-m}{m}n+2s_1,\ \mu_2 > \frac{2-m}{m}n+2\tilde s_2,$$

$$p > \frac{2+2\tilde{s}_2}{\mu_2} + \frac{2-m}{\mu_2 m}n,\ q > \frac{(2+2\tilde{s}_2)m}{n} + \frac{m}{2} + \frac{n\mu_2}{2m} \quad \textit{if} \quad \mu_1 > \frac{2-m}{m}n + 2s_1,\ \mu_2 < \frac{2-m}{m}n,$$

$$p > \frac{\mu_1}{\mu_2} + \frac{2+2\tilde{s}_2}{\mu_2},\ q > \frac{\mu_2}{\mu_1} + \frac{2+2s_1}{\mu_1} \quad \textit{if} \quad \max\{\mu_1;\mu_2\} < \frac{2-m}{m}n.$$

Then, there exists a uniquely determined globally (in time) energy solution to (4.1) in

$$\big(\mathcal{C}([0,\infty), H^{s_1}(\mathbb{R}^n)) \cap \mathcal{C}^1([0,\infty), H^{s_1-1}(\mathbb{R}^n))\big) \times \big(\mathcal{C}([0,\infty), H^{\tilde{s}_2}(\mathbb{R}^n)) \cap \mathcal{C}^1([0,\infty), H^{\tilde{s}_2-1}(\mathbb{R}^n))\big)\big).$$

Furthermore, the solution satisfies the following decay estimates:

$$\|u(t,\cdot)\|_{L^2(\mathbb{R}^n)} \lesssim \big(\|(u_0,u_1)\|_{\mathcal{A}_{m,s_1}} + \|(v_0,v_1)\|_{\mathcal{A}_{m,s_2}}\big) \times \begin{cases} (1+t)^{-\frac{2-m}{2m}n} & \textit{if} \quad \mu_1 > \frac{2-m}{m}n + 2s_1, \\ (1+t)^{-\frac{\mu_1}{2}} & \textit{if} \quad \mu_1 < \frac{2-m}{m}n, \end{cases}$$

$$\|u_t(t,\cdot)\|_{L^2(\mathbb{R}^n)} \lesssim \big(\|(u_0,u_1)\|_{\mathcal{A}_{m,s_1}} + \|(v_0,v_1)\|_{\mathcal{A}_{m,s_2}}\big) \times \begin{cases} (1+t)^{-\frac{2-m}{2m}n-1} & \textit{if} \quad \mu_1 > \frac{2-m}{m}n + 2s_1, \\ (1+t)^{-\frac{\mu_1}{2}} & \textit{if} \quad \mu_1 < \frac{2-m}{m}n, \end{cases}$$

$$\||D|^{s_1}u(\tau,\cdot)\|_{L^2(\mathbb{R}^n)} + \||D|^{s_1-1}u_t(\tau,\cdot)\|_{L^2(\mathbb{R}^n)} \lesssim \big(\|(u_0,u_1)\|_{\mathcal{A}_{m,s_1}} + \|(v_0,v_1)\|_{\mathcal{A}_{m,s_2}}\big) \times \begin{cases} (1+t)^{-\frac{2-m}{2m}n-s_1} & \textit{if} \quad \mu_1 > \frac{2-m}{m}n + 2s_1, \\ (1+t)^{-\frac{\mu_1}{2}} & \textit{if} \quad \mu_1 < \frac{2-m}{m}n, \end{cases}$$

$$\|v(t,\cdot)\|_{L^2(\mathbb{R}^n)} \lesssim \big(\|(u_0,u_1)\|_{\mathcal{A}_{m,s_1}} + \|(v_0,v_1)\|_{\mathcal{A}_{m,s_2}}\big) \times \begin{cases} (1+t)^{-\frac{2-m}{2m}n} & \textit{if} \quad \mu_2 > \frac{2-m}{m}n + 2\tilde{s}_2, \\ (1+t)^{-\frac{\mu_2}{2}} & \textit{if} \quad \mu_2 < \frac{2-m}{m}n, \end{cases}$$

$$\|v_t(t,\cdot)\|_{L^2(\mathbb{R}^n)} \lesssim \big(\|(u_0,u_1)\|_{\mathcal{A}_{m,s_1}} + \|(v_0,v_1)\|_{\mathcal{A}_{m,s_2}}\big) \times \begin{cases} (1+t)^{-\frac{2-m}{2m}n-1} & \textit{if} \quad \mu_2 > \frac{2-m}{m}n + 2\tilde{s}_2, \\ (1+t)^{-\frac{\mu_2}{2}} & \textit{if} \quad \mu_2 < \frac{2-m}{m}n, \end{cases}$$

$$\||D|^{\tilde{s}_2}v(\tau,\cdot)\|_{L^2(\mathbb{R}^n)} + \||D|^{\tilde{s}_2-1}u_t(\tau,\cdot)\|_{L^2(\mathbb{R}^n)} \lesssim \big(\|(u_0,u_1)\|_{\mathcal{A}_{m,s_1}} + \|(v_0,v_1)\|_{\mathcal{A}_{m,s_2}}\big) \times \begin{cases} (1+t)^{-\frac{2-m}{2m}n-\tilde{s}_2} & \textit{if} \quad \mu_2 > \frac{2-m}{m}n + 2\tilde{s}_2, \\ (1+t)^{-\frac{\mu_2}{2}} & \textit{if} \quad \mu_2 < \frac{2-m}{m}n. \end{cases}$$

Proof. The proof can be obtained by following the same steps of the proof of Theorem 4.5, but using the estimates of Lemma 4.3 instead of the estimates of Lemma 4.2. In other words, we use the rules for fractional powers instead of the fractional chain rule. □

4.5. Effective case versus scale-invariant case

Let us consider the following weakly coupled systems of semilinear damped wave equations which are studied in the last two chapters:

Effective case	Scale-invariant case
$u_{tt} - \Delta u + \frac{1}{(1+t)^{r_1}} u_t = \|v\|^p,$	$u_{tt} - \Delta u + \frac{\mu_1}{1+t} u_t = \|v\|^p,$
$v_{tt} - \Delta v + \frac{1}{(1+t)^{r_2}} v_t = \|u\|^q,$	$v_{tt} - \Delta v + \frac{\mu_2}{1+t} v_t = \|u\|^q,$
$u(0,x) = u_0(x),\ u_t(0,x) = u_1(x),$	$u(0,x) = u_0(x),\ u_t(0,x) = u_1(x),$
$v(0,x) = v_0(x),\ v_t(0,x) = v_1(x),$	$v(0,x) = v_0(x),\ v_t(0,x) = v_1(x),$

where $r_1, r_2 \in (-1,1)$ and μ_1, μ_2 are positive real numbers. Let us compare the obtained results for both models from two different points of view. On the one hand we are interested to compare the results from the point of view of the regularity of the data which is supposed. On the other hand we compare the results from the point of view of the values of μ_1 and μ_2.

4.5.1. Data from energy space:

First, let us assume that the values of μ_1 and μ_2 are sufficiently large. Comparing the statements of Theorem 4.4 and Theorem 3.4 we remark that the obtained results for both models coincide. In particular, the admissible ranges for the exponents p and q or for the modified exponents $\tilde{p}$ and $\tilde{q}$ of the power nonlinearities are the same. For this reason, we may interpret the above system for large values of μ_1 and μ_2 as a semilinear weakly coupled system of effectively damped wave equations. A similar remark was given in the case of a single equation in [6, 64, 65] and [71].
If μ_1 and μ_2 are positive and small, then we obtain quite different results for the above models. In particular, the admissible ranges for the exponents p and q presented in Theorem 4.4 are $p > \frac{\mu_1}{\mu_2} + \frac{4}{\mu_2}$ and $q > \frac{\mu_2}{\mu_1} + \frac{4}{\mu_1}$ which are quite different to the modified Fujita exponent $p_{Fuj,m}(n)$ for the effectively damped case. We see, that the results are more restrictive for the scale-invariant case. For this reason, we should interpret the above system for small μ_1 and μ_2 as a semilinear weakly coupled system of non-effectively damped wave equations.

4.5.2. Data with high regularity:

Now the data is supposed to belong to a Sobolev space with suitable regularity which is even embedded in $L^\infty(\mathbb{R}^n)$. If μ_1 and μ_2 are large, then from Theorem 3.4 of Chapter 3 and Theorem 4.6 of Chapter 4 one may conclude that the admissible range of the exponents of power nonlinearities coincide for both models. In this case lower bounds for the admissible ranges for the exponents p and q or for the modified

exponents $\tilde{p}$ and $\tilde{q}$ of the power nonlinearities are determined by the regularity parameter s.
Similarly to the case where the data are assumed from the energy space we can consider the system for large μ_1 and μ_2 as a semilinear weakly coupled system of effectively damped wave equations.
If μ_1 and μ_2 are positive and small, then we feel among other parameters a weak influence of the regularity parameters s_1 and s_2 on the admissible ranges of exponents which are determined as follow: $p > \frac{\mu_1}{\mu_2} + \frac{2+2s_2}{\mu_2}$ and $q > \frac{\mu_2}{\mu_1} + \frac{2+2s_1}{\mu_1}$. Then the results for the scale-invariant case are different than those for the effective case.

In a forthcoming project, we would like to consider the above scale-invariant model, where one of the positive values of μ_1 and μ_2 is large, and the other one is supposed to be small, let us say μ_1 is large and μ_2 is small. We want to understand under which assumptions the small value of μ_2 has no dominant influence. Another model of interest was proposed by Prof. Nakao (Kyushu University). It is the following model:

$$\begin{aligned} &u_{tt} - \Delta u + u_t = |v|^p, \qquad v_{tt} - \Delta v = |u|^q \\ &u(0,x) = u_0(x),\ u_t(0,x) = u_1(x),\ \ v(0,x) = v_0(x),\ v_t(0,x) = v_1(x). \end{aligned}$$

To understand the solvability behavior for this model is a big challenge and would hint to expectations for models with effective and non-effective dissipation terms in the sense of Wirth.

5. Blow up results for semilinear systems of weakly coupled effectively damped waves

5.1. Test function method

The sharpness of the results for the global (in time) existence of small data solutions or the notion of "blow-up of local (in time) solutions" means that if the pivotal condition for the global (in time) existence is not satisfied, then the solution does, in general, not exist globally (in time). Among several methods to prove blow-up results, the test function method is an important method which was introduced by Zhang in the paper [72].

Let us consider the Cauchy problem for the classical damped wave equation with power nonlinearity

$$u_{tt} - \Delta u + u_t = |u|^p, \;\; u(0,x) = u_0(x), \;\; u_t(0,x) = u_1(x), \tag{5.1}$$

where $(t,x) \in [0,\infty) \times \mathbb{R}^n$.

The nonexistence result for $p = p_{Fuj}(n)$ has been established in [72]. Todorova and Yordanov proved in [63] that $p_{Fuj}(n) = 1 + \frac{2}{n}$ is critical.

In the following we will recall and sketch the proof of a result which the reader can find in the book of Ebert and Reissig [13]. In the proof of Theorem 5.1 we explain the basics and the philosophy of the test function method.

Theorem 5.1. *Let $(u_0,u_1) \in \mathcal{A}_{1,1} = (H^1 \cap L^1) \times (L^2 \cap L^1)$ satisfy the assumption*

$$\int_{\mathbb{R}^n} \big(u_0(x) + u_1(x)\big) dx > 0, \tag{5.2}$$

with $n \geq 1$ and $p \in (1, 1 + \frac{2}{n}]$. Then there exists a locally (in time) defined energy solution u in $\mathcal{C}([0,\infty), H^1) \cap \mathcal{C}^1([0,\infty), L^2)$. This solution cannot be continued to the interval $[0,\infty)$ in time.

Proof. First we introduce the following test functions $\eta = \eta(t)$ and $\theta = \theta(x)$ having the following properties:

1. $\eta \in \mathcal{C}_0^\infty[0,\infty)$, $0 \leq \eta(t) \leq 1$,

$$\eta(t) = \begin{cases} 1 \text{ for } 0 \leq t \leq \frac{1}{2}, \\ 0 \text{ for } t \geq 1, \end{cases}$$

2. $\phi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$, $0 \le \phi(x) \le 1$,

$$\phi(x) = \begin{cases} 1 \text{ for } 0 \le |x| \le \frac{1}{2}, \\ 0 \text{ for } |x| \ge 1, \end{cases}$$

3. $\frac{\eta'(t)^2}{\eta(t)} \le C$ for $\frac{1}{2} < t < 1$, and $\frac{|\nabla\phi(x)|^2}{\phi(x)} \le C$ for $\frac{1}{2} < |x| < 1$.

Let R be a large parameter in $[0, \infty)$. We define the test function

$$\psi_R(t,x) := \eta_R(t)\phi_R(x) = \eta\Big(\frac{t}{R^2}\Big)\phi\Big(\frac{x}{R}\Big). \tag{5.3}$$

We put

$$Q_R := [0, R^2] \times B_R \text{ with } B_R := \{x \in \mathbb{R}^n : |x| \le R\}.$$

We note that the support of ψ_R is contained in the set Q_R. Moreover, $\psi_R \equiv 1$ on $[0, \frac{R^2}{2}] \times B_{\frac{R}{2}}$. We suppose that the energy solution $u = u(t,x)$ exists globally in time. We define the functional

$$I_R := \int_{Q_R} |u(t,x)|^p \psi_R^{p'}(t,x)\, d(t,x) = \int_{Q_R} (u_{tt} - \Delta u + u_t)\psi_R^{p'}(t,x)\, d(t,x),$$

where $\frac{1}{p'} + \frac{1}{p} = 1$. After integration by parts and using the condition $1 < p \le \frac{2}{n} + 1$ we obtain

$$I_R \lesssim I_R^{\frac{1}{p}},$$

which implies that $\int_0^\infty \int_{\mathbb{R}^n} |u(t,x)|^p\, d(t,x) = 0$. This contradicts the assumption (5.2). The reader can find a detailed proof in [13]. □

The Cauchy problem (5.1) has also been investigated by many authors [20, 21, 22, 23, 25, 27, 28, 29, 32, 33, 36, 41, 42, 43, 44, 72, 73].
Let us now consider the weakly coupled system of semilinear classical damped waves

$$\begin{aligned} &u_{tt} - \Delta u + u_t = |v|^p, \quad v_{tt} - \Delta v + v_t = |u|^q, \\ &u(0,x) = u_0(x),\ u_t(0,x) = u_1(x),\ v(0,x) = v_0(x),\ v_t(0,x) = v_1(x), \end{aligned} \tag{5.4}$$

where $(t,x) \in [0,\infty) \times \mathbb{R}^n$, $p, q \ge 1$ and $pq > 1$. Motivated by some previous papers concerned with the case of the Cauchy problem for a semilinear single equation, the authors in [45] and [60] studied the blow-up behavior of solutions of the system (5.4). In the following theorem we will recall and prove the result of F. Sun and M. Wang published in [60].

Theorem 5.2. *Let $n \ge 1$. Assume that $q \ge p \ge 1$ and $\frac{n}{2} \le \frac{q+1}{pq-1}$. If the data satisfy*

$$(u_i, v_i) \in [W^{1-i,1}(\mathbb{R}^n) \cap W^{1-i,\infty}(\mathbb{R}^n)]^2 \text{ for } i = 0, 1,$$

and

$$\int_{\mathbb{R}^n} u_i(x)dx > 0, \qquad \int_{\mathbb{R}^n} v_i(x)dx > 0 \text{ for } i = 0, 1, \tag{5.5}$$

then the Sobolev solution (u, v) of the Cauchy problem (5.4) does not exist globally (in time).

Proof. We assume that the Sobolev solution (u, v) exist globally (in time). Let $\phi \in \mathcal{C}_0^2(\mathbb{R}, [0,\infty))$ have the following properties:

$$\phi(r) = \begin{cases} 1 & \text{if} \quad r \leq 1, \\ 0 & \text{if} \quad r \geq 2, \end{cases} \tag{5.6}$$

where $0 \leq \phi(r) \leq 1$, and $r|\phi'(r)| \leq C$ for any $r > 0$.
We choose the test function $\xi(t,x) = \phi^\lambda\big(\frac{t^2+|x|^4}{R^4}\big)$, where $\lambda \gg 1$ and R are large parameters in $[0,\infty)$. Similar to the treatment of one equation (5.1) we get for the above equations

$$\begin{aligned} \int_0^\infty \int_{\mathbb{R}^n} |v|^p \xi dxdt \qquad &= -\big(\int_{\mathbb{R}^n} (u_0(x) + u_1(x))\xi(0,x)dx\big) \\ &\quad + \int_0^\infty \int_{\mathbb{R}^n} u(\xi_t + \xi_{tt} - \Delta\xi)dxdt, \end{aligned} \tag{5.7}$$

$$\begin{aligned} \int_0^\infty \int_{\mathbb{R}^n} |u|^q \xi dxdt \qquad &= -\big(\int_{\mathbb{R}^n} (v_0(x) + v_1(x))\xi(0,x)dx\big) \\ &\quad + \int_0^\infty \int_{\mathbb{R}^n} v(\xi_t + \xi_{tt} - \Delta\xi)dxdt. \end{aligned} \tag{5.8}$$

From (5.7), we get

$$\begin{aligned} \Big| \int_0^\infty & \int_{\mathbb{R}^n} |v|^p \xi dxdt \Big| \\ &\leq \int_0^\infty \int_{\mathbb{R}^n} |u|\, |\xi_t + \xi_{tt} - \Delta\xi|\, dxdt \\ &\leq \int_0^\infty \int_{\mathbb{R}^n} |u||\xi_t| dxdt + \int_0^\infty \int_{\mathbb{R}^n} |u||\xi_{tt}| dxdt + \int_0^\infty \int_{\mathbb{R}^n} |u||\Delta\xi| dxdt \\ &= \int_0^\infty \int_{\mathbb{R}^n} \big(|u|\xi^{\frac{1}{q}}\big)\big(\xi^{-\frac{1}{q}}|\xi_t|\big) dxdt + \int_0^\infty \int_{\mathbb{R}^n} \big(|u|\xi^{\frac{1}{q}}\big)\big(\xi^{-\frac{1}{q}}|\xi_{tt}|\big) dxdt \\ &\quad + \int_0^\infty \int_{\mathbb{R}^n} \big(|u|\xi^{\frac{1}{q}}\big)\big(\xi^{-\frac{1}{q}}|\Delta\xi|\big) dxdt \\ &\leq \Big(\int_0^\infty \int_{\mathbb{R}^n} |u|^q \xi dxdt\Big)^{\frac{1}{q}} \Big(\int_0^\infty \int_{\mathbb{R}^n} \xi^{-\frac{q'}{q}} |\xi_t|^{q'}\Big)^{\frac{1}{q'}} \\ &\quad + \Big(\int_0^\infty \int_{\mathbb{R}^n} |u|^q \xi dxdt\Big)^{\frac{1}{q}} \Big(\int_0^\infty \int_{\mathbb{R}^n} \xi^{-\frac{q'}{q}} |\xi_{tt}|^{q'}\Big)^{\frac{1}{q'}} \\ &\quad + \Big(\int_0^\infty \int_{\mathbb{R}^n} |u|^q \xi dxdt\Big)^{\frac{1}{q}} \Big(\int_0^\infty \int_{\mathbb{R}^n} \xi^{-\frac{q'}{q}} |\Delta\xi|^{q'}\Big)^{\frac{1}{q'}} \\ &\leq \Big(\int_0^\infty \int_{\mathbb{R}^n} |u|^q \xi dxdt\Big)^{\frac{1}{q}} A_q, \end{aligned}$$

where $\frac{1}{q} + \frac{1}{q'} = 1$ and

$$\begin{aligned} A_q &= \Big(\int_0^\infty \int_{\mathbb{R}^n} \xi^{-\frac{q'}{q}} |\xi_t|^{q'}\Big)^{\frac{1}{q'}} + \Big(\int_0^\infty \int_{\mathbb{R}^n} \xi^{-\frac{q'}{q}} |\xi_{tt}|^{q'}\Big)^{\frac{1}{q'}} + \Big(\int_0^\infty \int_{\mathbb{R}^n} \xi^{-\frac{q'}{q}} |\Delta\xi|^{q'}\Big)^{\frac{1}{q'}} \\ &= \Big(\int_{\sum} \xi^{-\frac{q'}{q}} |\xi_t|^{q'}\Big)^{\frac{1}{q'}} + \Big(\int_{\sum} \xi^{-\frac{q'}{q}} |\xi_{tt}|^{q'}\Big)^{\frac{1}{q'}} + \Big(\int_{\sum} \xi^{-\frac{q'}{q}} |\Delta\xi|^{q'}\Big)^{\frac{1}{q'}}, \end{aligned}$$

where

$$\sum = \big\{(t,x) \in [0,\infty) \times \mathbb{R}^n : R^4 < t^2 + |x|^4 < 2R^4\big\}.$$

In the same way we can prove

$$\int_0^\infty \int_{\mathbb{R}^n} |u|^q \xi dxdt \leq \Big(\int_0^\infty \int_{\mathbb{R}^n} |v|^p \xi dxdt \Big)^{\frac{1}{p}} A_p,$$

where $\frac{1}{p} + \frac{1}{p'} = 1$ and

$$A_p = \Big(\int_\Sigma \xi^{-\frac{p'}{p}} |\xi_t|^{p'} \Big)^{\frac{1}{p'}} + \Big(\int_\Sigma \xi^{-\frac{p'}{p}} |\xi_{tt}|^{p'} \Big)^{\frac{1}{p'}} + \Big(\int_\Sigma \xi^{-\frac{p'}{p}} |\Delta \xi|^{p'} \Big)^{\frac{1}{p'}}.$$

Summarizing, we get

$$\int_0^\infty \int_{\mathbb{R}^n} |v|^p \xi dxdt \leq A_q \Big(\int_0^\infty \int_{\mathbb{R}^n} |u|^q \xi dxdt \Big)^{\frac{1}{q}}, \tag{5.9}$$

$$\int_0^\infty \int_{\mathbb{R}^n} |u|^q \xi dxdt \leq A_p \Big(\int_0^\infty \int_{\mathbb{R}^n} |v|^p \xi dxdt \Big)^{\frac{1}{p}}. \tag{5.10}$$

From (5.9) and (5.10) we obtain

$$\int_0^\infty \int_{\mathbb{R}^n} |v|^p \xi dxdt \leq A_q (A_p)^{\frac{1}{q}} \Big(\int_0^\infty \int_{\mathbb{R}^n} |v|^p \xi dxdt \Big)^{\frac{1}{pq}},$$

and

$$\int_0^\infty \int_{\mathbb{R}^n} |u|^q \xi dxdt \leq A_p (A_q)^{\frac{1}{p}} \Big(\int_0^\infty \int_{\mathbb{R}^n} |u|^q \xi dxdt \Big)^{\frac{1}{pq}}.$$

Then,

$$\Big(\int_0^\infty \int_{\mathbb{R}^n} |v|^p \xi dxdt \Big)^{\frac{pq-1}{pq}} \leq A_q (A_p)^{\frac{1}{q}}, \tag{5.11}$$

$$\Big(\int_0^\infty \int_{\mathbb{R}^n} |u|^q \xi dxdt \Big)^{\frac{pq-1}{pq}} \leq A_p (A_q)^{\frac{1}{p}}. \tag{5.12}$$

Now to compute the right-hand sides of (5.11) and (5.12) we apply the change of variables $t = R^2 \tau$ and $x = ry$. Then we have

$$dtdx = R^{n+2} dyd\tau; \qquad \xi_t = R^{-2} \xi_\tau; \qquad \xi_{tt} = R^{-4} \xi_\tau; \qquad \Delta \xi = R^{-2} \delta \xi_\tau;$$

$$R^4 \leq t^2 + |x|^4 \leq 2R^2 \Rightarrow 1 \leq \tau^2 + |y|^4 \leq 2.$$

We obtain

$$\begin{aligned} A_p &\lesssim R^{\left(n+2-\frac{2p}{p-1}\right)\frac{p-1}{p}} + R^{\left(n+2-\frac{4p}{p-1}\right)\frac{p-1}{p}} + R^{\left(n+2-\frac{2p}{p-1}\right)\frac{p-1}{p}} \lesssim R^{\left(n+2-\frac{2p}{p-1}\right)\frac{p-1}{p}}, \\ A_q &\lesssim R^{\left(n+2-\frac{2q}{q-1}\right)\frac{q-1}{q}} + R^{\left(n+2-\frac{4q}{q-1}\right)\frac{q-1}{q}} + R^{\left(n+2-\frac{2q}{q-1}\right)\frac{q-1}{q}} \lesssim R^{\left(n+2-\frac{2q}{q-1}\right)\frac{q-1}{q}}. \end{aligned}$$

Using (5.11) and (5.12) we get

$$\Big(\int_0^\infty \int_{\mathbb{R}^n} |v|^p \xi dxdt \Big)^{\frac{pq-1}{pq}} \lesssim R^{s_1}, \tag{5.13}$$

$$\Big(\int_0^\infty \int_{\mathbb{R}^n} |u|^q \xi dxdt \Big)^{\frac{pq-1}{pq}} \lesssim R^{s_2}, \tag{5.14}$$

where
$$s_1 = \Big(n+2-\frac{2q}{q-1}\Big)\frac{q-1}{q} + \Big(n+2-\frac{2p}{p-1}\Big)\frac{p-1}{pq},$$
and
$$s_2 = \Big(n+2-\frac{2p}{p-1}\Big)\frac{p-1}{p} + \Big(n+2-\frac{2q}{q-1}\Big)\frac{q-1}{pq}.$$
We have $s_1 \leq 0$ from $q \geq p > 1$, and $s_2 \leq 0$ if and only if $\frac{n}{2} \leq \frac{q+1}{pq-1}$. For s_2 we distinguish two cases.

1. $s_2 < 0$: For $R \to \infty$, from (5.14) we have
$$\int_0^\infty \int_{\mathbb{R}^n} |u|^q \xi dx dt = 0$$
which implies $(u,v) = (0,0)$. This contradicts our assumptions.

2. $s_2 = 0$: In this case there is positive constant C independent of R such that
$$\int_0^\infty \int_{\mathbb{R}^n} |u|^q \xi dx dt \leq C,$$
which implies that
$$\int_0^\infty \int_{\sum} |u|^q \xi dx dt \to 0 \qquad \text{as} \qquad R \to \infty.$$
Previously we have
$$\int_0^\infty \int_{\mathbb{R}^n} |u|^q \xi dx dt \lesssim A_p(A_q)^{\frac{1}{p}} \Big(\int_0^\infty \int_{\sum} |u|^q \xi dx dt\Big)^{\frac{1}{pq}} \lesssim R^{s_2} \Big(\int_0^\infty \int_{\sum} |u|^q \xi dx dt\Big)^{\frac{1}{pq}}.$$
Letting $R \to \infty$ we get the same contradiction.

This completes the proof. □

5.2. Blow-up result for weakly coupled systems of semilinear damped waves with different coefficients in the dissipation terms

Firstly, let us consider the Cauchy problem for a semilinear classical damped wave equation, namely
$$u_{tt} - \Delta u + b(t)u_t = |u|^p, \;\; u(0,x) = u_0(x), \;\; u_t(0,x) = u_1(x), \tag{5.15}$$
where the dissipation term $b(t)u_t$ is supposed to be effective in the sense of Wirth. In [4] the authors determined the critical exponent $p = p_{Fuj}(n)$. That means after proving

the global existence for some admissible range $p > p_{Fuj}(n)$, the authors proved also that, in general, the solution cannot be globally defined for $1 < p \leq p_{Fuj}(n)$. In other words, we have, in general, only local solutions (in time). The case $b(t) = \frac{\mu}{(1+t)^r}$ with $\mu > 0$ and $r > 0$ was studied in [37].
Let us consider now the Cauchy problem for the following system:

$$\begin{aligned} &u_{tt} - \Delta u + b(t)u_t = |v|^p, \qquad v_{tt} - \Delta v + b(t)v_t = |u|^q, \\ &u(0,x) = u_0(x),\;\; u_t(0,x) = u_1(x),\;\; v(0,x) = v_0(x),\;\; v_t(0,x) = v_1(x), \end{aligned} \tag{5.16}$$

where $(t,x) \in [0,\infty) \times \mathbb{R}^n$. As we already remarked during the treatment of the models (5.4) and (5.1) the test function method is not influenced by higher regularity of the data. We restrict ourselves to prove the sharpness of our results for the Cauchy problem (5.16), where the data are supposed to belong to the energy space. In the following we will prove the optimality of our results from Theorem 2.10. That means, if

$$\frac{n}{2} < \frac{\max\{p;q\}+1}{pq-1},$$

then, in general, the local (in time) energy solution cannot be extended globally. The ideas of the proof of the following theorem are based on the paper [9] which is devoted to study a general case of model (5.15).

Theorem 5.3. *Let $b = b(t)$ satisfy the conditions of Hypothesis 1.1. Moreover, let*

$$\liminf_{t\longrightarrow\infty} \frac{b'(t)}{b(t)^2} > -1,\;\; \limsup_{t\longrightarrow\infty} \frac{tb'(t)}{b(t)} < 1$$

and let

$$\frac{n}{2} \leq \frac{q+1}{pq-1},$$

where $q > p > 1$ and $pq > 1$. Then there exists no global classical solution $(u,v) \in \left(\mathcal{C}^2([0,\infty) \times \mathbb{R}^n)\right)^2$ to (5.16) with initial data $((u_0,u_1),(v_0,v_1)) \in \mathcal{A}_{1,1} \times \mathcal{A}_{1,1}$ such that

$$\int_{\mathbb{R}^n} u_0(x) + \hat{b}_1^{-1} u_0(x)dx > 0,$$

$$\int_{\mathbb{R}^n} v_0(x) + \hat{b}_1^{-1} v_0(x)dx > 0.$$

Before proving this theorem we show the following lemma which will be used later in the proof. The proof of Lemma 5.1 can be concluded from [9] and [37].

Lemma 5.1. *Let $g = g(t) \in \mathcal{C}([0,\infty))$ be a solution of the following initial value problem for an ordinary differential equation:*

$$-g'(t) + g(t)b(t) = 1,\;\; g(0) = \frac{1}{\hat{b}_1}.$$

If $b = b(t)$ satisfies the assumptions of Theorem 5.3 , then it holds $g(t) \approx \frac{1}{b(t)}$ and

$$|g'(t) - 1| \leq C. \tag{5.17}$$

Proof. We multiply (5.16) by the positive function $g = g(t)$ which is defined in Lemma 5.1. In this way we obtain

$$\begin{aligned} (g(t)u)_{tt} - \Delta(g(t)u) - (g'(t)u)_t + (-g'(t) + g(t)b(t))u_t &= g(t)|v|^p, \\ (g(t)v)_{tt} - \Delta(g(t)v) - (g'(t)v)_t + (-g'(t) + g(t)b(t))v_t &= g(t)|u|^a. \end{aligned}$$

From the definition of $g = g(t)$ we may conclude

$$\begin{aligned} (g(t)u)_{tt} - \Delta(g(t)u) - (g'(t)u)_t + u_t = g(t)|v|^p, \\ (g(t)v)_{tt} - \Delta(g(t)v) - (g'(t)v)_t + v_t = g(t)|u|^q. \end{aligned}$$

We introduce the test functions
$\eta \in \mathcal{C}_0^\infty[0,\infty)$ with $0 \leq \eta(t) \leq 1$, where

$$\eta(t) = \begin{cases} 1 \text{ for } 0 \leq t \leq \frac{1}{2}, \\ 0 \text{ for } t \geq 1, \end{cases}$$

$\phi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ with $0 \leq \phi(x) \leq 1$, where

$$\phi(x) = \begin{cases} 1 \text{ for } 0 \leq |x| \leq \frac{1}{2}, \\ 0 \text{ for } |x| \geq 1. \end{cases}$$

Moreover, we assume

$$\max\Big\{\frac{|\eta'(t)|^\beta}{\eta(t)}; \frac{|\eta''(t)|^\alpha}{\eta(t)}\Big\} \leq C \text{ for } \frac{1}{2} \leq t \leq 1,$$

and

$$\max\Big\{\frac{|\bigtriangledown \phi(x)|^{\beta'}}{\phi(x)}; \frac{|\Delta\phi(x)|^{\alpha'}}{\phi(x)}\Big\} \leq C \text{ for } \frac{1}{2} < |x| < 1,$$

where we choose $1 < \alpha, \beta, \alpha', \beta' < \min\{p; q\}$. Let R be a large parameter in $[0,\infty)$ and

$$Q_R := [0, F(R)] \times B_R, \;\; B_R := \{x \in \mathbb{R}^n, |x| \leq R\}.$$

We define the test function

$$\psi_R(t,x) := \eta_R(t)\phi_R(x) = \eta\Big(\frac{t}{F(R)}\Big)\phi\Big(\frac{x}{R}\Big),$$

where $F(R) = B^{-1}(R^2, 0)$ and $B^{-1}(t,0)$ is the inverse function of $B(t,0)$. It follows that
$F : [0,\infty) \longrightarrow [0,\infty)$ is a strictly increasing function with $F(0) = 0$ and $\lim\limits_{R\longrightarrow\infty} F(R) = \infty$. Moreover, we have $R \lesssim F(R)$ as a result of $b(t) \gtrsim (1+t)^{-1}$.
We have after integrating by parts

$$\begin{aligned} \int_{Q_R} g(t)|v|^p\psi_R d(t,x) &= -\int_{B_R} (u_0 + \hat{b}_1^{-1}u_1)\phi_R^\lambda dx \\ &+ \int_{Q_R} \big(g(t)u\partial_t^2\psi_R + (g'(t)-1)u\partial_t\psi_R + g(t)u\Delta\psi_R\big)d(t,x), \end{aligned}$$

and

$$\begin{aligned}\int_{Q_R} g(t)|u|^q\psi_R d(t,x) &= -\int_{B_R}(v_0+\hat{b}_1^{-1}v_1)\phi_R^\lambda dx \\ &\quad+\int_{Q_R}\big(g(t)v\partial_t^2\psi_R+(g'(t)-1)v\partial_t\psi_R+g(t)v\Delta\psi_R\big)d(t,x).\end{aligned}$$

This implies

$$\int_{Q_R} g(t)|v|^p\psi_R d(t,x) \lesssim \int_{Q_R}\big(g(t)u\partial_t^2\psi_R+(g'(t)-1)u\partial_t\psi_R+g(t)u\Delta\psi_R\big)d(t,x),$$

and

$$\int_{Q_R} g(t)|u|^q\psi_R d(t,x) \lesssim \int_{Q_R}\big(g(t)v\partial_t^2\psi_R+(g'(t)-1)v\partial_t\psi_R+g(t)v\Delta\psi_R\big)d(t,x).$$

Using Lemma 5.1, Hölder's inequality with $\frac{1}{q}+\frac{1}{q'}=1$ and (5.17) we get

$$\begin{aligned}&\textstyle\int_{Q_R}|ug(t)\partial_t^2\psi_R|d(t,x) \\ &\quad\le\Big(\textstyle\int_{Q_R}|u|^q g(t)\psi_R d(t,x)\Big)^{\frac{1}{q}}\Big(\int_{Q_R}\psi_R^{-\frac{q'}{q}}g(t)|\partial_t^2\psi_R|^{q'}d(t,x)\Big)^{\frac{1}{q'}},\end{aligned} \tag{5.18}$$

$$\begin{aligned}&\textstyle\int_{Q_R}|u(g'(t)-1)\partial_t\psi_R|d(t,x) \\ &\quad\le\Big(\textstyle\int_{Q_R}|u|^q g(t)\psi_R d(t,x)\Big)^{\frac{1}{q}}\Big(\int_{Q_R}g(t)b(t)^{q'}\psi_R^{-\frac{q'}{q}}|\partial_t\psi_R|^{q'}d(t,x)\Big)^{\frac{1}{q'}},\end{aligned} \tag{5.19}$$

$$\begin{aligned}&\textstyle\int_{Q_R}|ug(t)\Delta\psi_R|d(t,x) \\ &\quad\le\Big(\textstyle\int_{Q_R}|u|^q g(t)\psi_R d(t,x)\Big)^{\frac{1}{q}}\Big(\int_{Q_R}\psi_R^{-\frac{q'}{q}}g(t)|\Delta\psi_R|^{q'}d(t,x)\Big)^{\frac{1}{q'}}.\end{aligned} \tag{5.20}$$

We apply a change of variables $t=F(R)\tau$ and $x=Ry$. Then we have

$$d(t,x)=F(R)R^n d(\tau,y),\quad \partial_t\psi_R=F(R)^{-1}\partial_\tau\psi_R,\quad \partial_t^2\psi_R=F(R)^{-2}\partial_\tau^2\psi_R,$$

$$\Delta\psi_R=R^{-2}\Delta\psi_R,\ \frac{F(R)}{2}\le t\le F(R),\ \frac{R}{2}\le|x|\le R\Longleftrightarrow\frac{1}{2}\le\tau,|y|\le 1.$$

With this change of variables we get for (5.18) the chain of inequalities

$$\begin{aligned}&\Big(\int_{Q_R}\psi_R^{-\frac{q'}{q}}g(t)|\partial_t^2\psi_R|^{q'}d(t,x)\Big)^{\frac{1}{q'}} \\ &\quad=\Big(\int_{\frac{1}{2}}^1\int_{\frac{1}{2}}^1\psi_R^{-\frac{q'}{q}}g(t)(F(R)\tau)F(R)^{-2q'}|\psi_R|^{\frac{q'}{\alpha}}F(R)R^n d\tau dy\Big)^{\frac{1}{q'}} \\ &\quad\lesssim\Big(F(R)^{-2q'}R^n\int_{\frac{F(R)}{2}}^{F(R)}g(t)dt\Big)^{\frac{1}{q'}} \\ &\quad\lesssim\Big(F(R)^{-2q'}R^n\int_{\frac{F(R)}{2}}^{F(R)}\frac{1}{b(t)}dt\Big)^{\frac{1}{q'}} \\ &\quad\lesssim\big(F(R)^{-2q'}R^nB(F(R))\big)^{\frac{1}{q'}} \\ &\quad\lesssim F(R)^{\frac{n+2-2q'}{q'}}.\end{aligned}$$

Consequently, we arrive at

$$\Big(\int_{Q_R} \psi_R^{-\frac{q'}{q}} g(t)|\partial_t^2\psi_R|^{q'} d(t,x)\Big)^{\frac{1}{q'}} \lesssim F(R)^{\frac{n+2-2q'}{q'}}. \tag{5.21}$$

In the same way we can prove for (5.20) the estimate

$$\Big(\int_{Q_R} \psi_R^{-\frac{q'}{q}} g(t)|\Delta\psi_R|^{q'} d(t,x)\Big)^{\frac{1}{q'}} \lesssim F(R)^{\frac{n+2-2q'}{q'}}. \tag{5.22}$$

Finally, let us turn to (5.19). We have

$$\begin{aligned}\Big(\int_{Q_R} g(t)b(t)^{q'}\psi_R^{-\frac{q'}{q}}|\partial_t\psi_R|^{q'} d(t,x)\Big)^{\frac{1}{q'}} &\lesssim \Big(F(R)^{-q'}\int_{Q_R} b(t)^{q'-1}\psi_R^{-\frac{q'}{q}}|\psi_R|^{\frac{q'}{\beta}} d(t,x)\Big)^{\frac{1}{q'}}\\ &\lesssim \Big(F(R)^{-q'}R^n\int_{\frac{F(R)}{2}}^{F(R)} b(t)^{q'-1}dt\Big)^{\frac{1}{q'}}.\end{aligned}$$

Since $F(0)=0$ and

$$F'(R)=(B^{-1}(R^2))'=\frac{2R}{B'(F(R))}=2Rb(F(R)),$$

by using $b(t)\approx b(\frac{t}{2})$ and $B(t,0)-B(\frac{t}{2},0)\approx B(t,0)$ we get

$$\int_{\frac{F(R)}{2}}^{F(R)} b(t)^{q'-1}dt \approx (b(F(R)))^{q'}\int_{\frac{F(R)}{2}}^{F(R)} b(t)^{-1}dt \approx (b(F(R)))^{q'}R^2.$$

Moreover, we have

$$\frac{b(F(R))}{F(R)} \approx \frac{1}{B(F(R),0)} = R^{-2}.$$

Finally, we obtain

$$\Big(\int_{Q_R} g(t)b(t)^{q'}\psi_R^{-\frac{q'}{q}}|\partial_t\psi_R|^{q'} d(t,x)\Big)^{\frac{1}{q'}} \lesssim F(R)^{\frac{n+2-2q'}{q'}}. \tag{5.23}$$

Consequently, from (5.21) to (5.23) we get

$$\int_{Q_R} g(t)|v|^p\psi_R d(t,x) \lesssim F(R)^{\frac{n+2-2q'}{q'}}\Big(\int_{Q_R}|u|^q g\psi_R d(t,x)\Big)^{\frac{1}{q}}. \tag{5.24}$$

On the contrary, one can get also

$$\int_{Q_R} g(t)|u|^q\psi_R d(t,x) \lesssim F(R)^{\frac{n+2-2p'}{p'}}\Big(\int_{Q_R}|v|^p g\psi_R d(t,x)\Big)^{\frac{1}{p}}. \tag{5.25}$$

From (5.24) and (5.25) we obtain

$$\Big(\int_{Q_R} g(t)|v|^p\psi_R d(t,x)\Big)^{\frac{pq-1}{pq}} \leq F(R)^{s_1}, \tag{5.26}$$

$$\Big(\int_{Q_R} g(t)|u|^q\psi_R d(t,x)\Big)^{\frac{pq-1}{pq}} \leq F(R)^{s_2}, \tag{5.27}$$

where

$$s_1 = \frac{n+2}{q'} - 2 + \Big(\frac{n+2}{p'} - 2\Big)\frac{1}{q} \quad \text{and} \quad s_2 = \frac{n+2}{p'} - 2 + \Big(\frac{n+2}{q'} - 2\Big)\frac{1}{p}.$$

The assumption $\frac{n}{2} \leq \frac{q+1}{pq-1}$ implies that $s_2 \leq 0$. We consider two cases.

- $s_2 < 0$: Letting $R \longrightarrow \infty$ in the inequality (5.27) we obtain

$$\int_0^\infty \int_{\mathbb{R}^n} g(t)|u|^q d(t,x) = 0.$$

This implies $u \equiv 0$. This is a contradiction to the assumptions.

- $s_2 = 0$: In this case we have

$$\begin{aligned} s_2 = 0 \quad &\Leftrightarrow \quad \frac{n}{2} = \frac{q+1}{pq-1} > \frac{p+1}{pq-1} \\ &\Leftrightarrow \quad s_1 < 0. \end{aligned}$$

Letting $R \longrightarrow \infty$ in the inequality (5.26) we obtain that

$$\int_0^\infty \int_{\mathbb{R}^n} g(t)|v|^q d(t,x) = 0.$$

This implies $v \equiv 0$. This is a contradiction to the assumptions.

The proof is complete. □

5.3. Concluding remarks

Let us now assume the following weakly coupled system of semilinear damped waves with different coefficients in the dissipation terms

$$\begin{aligned} &u_{tt} - \Delta u + \tfrac{1}{(1+t)^{r_1}} u_t = |v|^p, \qquad v_{tt} - \Delta v + \tfrac{1}{(1+t)^{r_2}} v_t = |u|^q, \\ &u(0,x) = u_0(x), \;\; u_t(0,x) = u_1(x), \;\; v(0,x) = v_0(x), \;\; v_t(0,x) = v_1(x), \end{aligned} \tag{5.28}$$

where the exponents $r_1, r_2 \in (-1,1)$. The global existence (in time) of solutions of this Cauchy problem was treated in Chapter 3, where the data are defined in different classes of regularity which are the followings: low regular data, data from energy space, data from Sobolev spaces with suitable regularity and, finally, large regular data. We mentioned and remarked in previous sections of this chapter that the regularity of the data did not appeared in the proofs of our results. For this reason we restrict ourselves in the near future to treat the blow up result for the Cauchy problem (5.28), where the data are assumed to belong to the energy space, in other words

$$(u_0,u_1),(v_0,v_1) \in \mathcal{A}_{1,1} \times \mathcal{A}_{1,1}.$$

From Chapter 3 one can see that the global (in time) existence depends strongly with the modified exponents of power nonlinearities $\tilde{p}, \tilde{q}$ and the exponents r_1, r_2 in the dissipation terms. Finally, we should prove the blow up in several cases with respect to the results of the summary table from Section 3.2. The blow up results which can be proved are described in the following table:

	$\tilde{p} < p_{Fuj}(n) < \tilde{q}$	$\tilde{q} < p_{Fuj}(n) < \tilde{p}$
$r_1 < r_2$ $\tilde{q} = \tilde{q}_{r_1,r_2,1}$ $\tilde{p} = \tilde{p}_{r_1,r_2}$	Blow up condition $\frac{n}{2} \leq \frac{\tilde{q}+1+\frac{1}{2}\left(\frac{r_1-r_2}{1+r_2}\right)}{\tilde{p}\tilde{q}-1+\frac{m}{2}(\tilde{p}-1)\left(\frac{r_1-r_2}{1+r_2}\right)}$	Blow up condition $\frac{n}{2} \leq \frac{\tilde{p}+\frac{1+r_2}{1+r_1}}{\tilde{p}\tilde{q}-\frac{1+r_2}{1+r_1}+\tilde{q}\left(\frac{r_2-r_1}{1+r_1}\right)}$
$r_2 < r_1$ $\tilde{q} = \tilde{q}_{r_1,r_2}$ $\tilde{p} = \tilde{p}_{r_1,r_2,1}$	Blow up condition $\frac{n}{2} \leq \frac{\tilde{q}+\frac{1+r_1}{1+r_2}}{\tilde{p}\tilde{q}-\frac{1+r_1}{1+r_2}+\tilde{p}\left(\frac{r_1-r_2}{1+r_2}\right)}$	Blow up condition $\frac{n}{2} \leq \frac{\tilde{p}+1+\frac{m}{2}\left(\frac{r_2-r_1}{1+r_1}\right)}{\tilde{p}\tilde{q}-1+\frac{1}{2}(\tilde{q}-1)\left(\frac{r_2-r_1}{1+r_1}\right)}$

A. Appendix

In the Appendix we collect some background material which is helpful and important for our research. Most of these tools are from the theory of harmonic analysis and function spaces. In particular, these tools allow us to estimate power nonlinearities in some scale of function spaces of different regularity.

A.1. Gagliardo-Nirenberg inequality

To control suitable norms of the nonlinear terms of our starting system or equation we shall apply several times Gagliardo-Nirenberg inequality or its generalization to the case of Sobolev spaces of fractional order or the so-called fractional Gagliardo-Nirenberg inequality which is stated in the following proposition.

Proposition A.1. *Let $1 < p, p_0, p_1 < \infty, \sigma > 0$ and $s \in [0, \sigma)$. Then it holds the following fractional Gagliardo-Nirenberg inequality for all $u \in L^{p_0}(\mathbb{R}^n) \cap \dot{H}^{\sigma}_{p_1}(\mathbb{R}^n)$:*

$$\|u\|_{\dot{H}^s_p(\mathbb{R}^n)} \lesssim \|u\|^{(1-\theta)}_{L^{p_0}(\mathbb{R}^n)} \|u\|^{\theta}_{\dot{H}^{\sigma}_{p_1}(\mathbb{R}^n)}, \tag{A.1}$$

where $\theta = \theta_{s,\sigma} := \frac{\frac{1}{p_0} - \frac{1}{p} + \frac{s}{n}}{\frac{1}{p_0} - \frac{1}{p_1} + \frac{\sigma}{n}}$ and $\frac{s}{\sigma} \leq \theta \leq 1$.

For the proof see [19] and [2, 16, 17, 18, 49, 50].

Corollary A.1. *Let $1 < p, m < \infty, \sigma > 0$ and $s \in [0, \sigma)$. Then we have the following inequality for all $u \in H^{\sigma}_m(\mathbb{R}^n)$:*

$$\||D|^s u\|_{L^p(\mathbb{R}^n)} \lesssim \|u\|^{(1-\theta)}_{L^m(\mathbb{R}^n)} \||D|^{\sigma} u\|^{\theta}_{L^m(\mathbb{R}^n)}, \tag{A.2}$$

where $\theta = \theta_{s,\sigma}(p, m) := \frac{n}{\sigma}\big(\frac{1}{m} - \frac{1}{p} + \frac{s}{n}\big)$ and $\frac{s}{\sigma} \leq \theta_{s,\sigma}(p, m) \leq 1$.

If $s = 0$ and $m = 2$ in Corollary A.1, then the fractional Gagliardo-Nirenberg inequality reduces to the **classical Gagliardo-Nirenberg inequality**

$$\|u\|_{L^p(\mathbb{R}^n)} \lesssim \|u\|^{(1-\theta)}_{L^2(\mathbb{R}^n)} \||D|^{\sigma} u\|^{\theta}_{L^2(\mathbb{R}^n)}, \tag{A.3}$$

where $\theta = \theta(p) := \frac{n}{\sigma}\big(\frac{1}{2} - \frac{1}{p}\big)$ and $0 \leq \theta(p) \leq 1$.

A.2. Fractional Leibniz rule

Proposition A.2. *Let us assume $s > 0$ and $1 \leq r \leq \infty, 1 < p_1, p_2, q_1, q_2 \leq \infty$ satisfying the following relation*

$$\frac{1}{r} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{q_1} + \frac{1}{q_2}.$$

Then the following fractional Leibniz rule holds:

$$\||D|^s(fg)\|_{L^r(\mathbb{R}^n)} \lesssim \||D|^s f\|_{L^{p_1}(\mathbb{R}^n)} \|g\|_{L^{p_2}(\mathbb{R}^n)} + \|f\|_{L^{q_1}(\mathbb{R}^n)} \||D|^s g\|_{L^{q_2}(\mathbb{R}^n)} \tag{A.4}$$

for all $f \in \dot{H}^s_{p_1}(\mathbb{R}^n) \cap L^{q_1}(\mathbb{R}^n)$ and $g \in \dot{H}^s_{q_2}(\mathbb{R}^n) \cap L^{p_2}(\mathbb{R}^n)$.

For the proof one can see [16].

A.3. Fractional chain rule

Proposition A.3. *Let us choose $s \in (0,1), 1 < r, r_1, r_2 < \infty$ and a $\mathcal{C}^1$ function F satisfying for any $\tau \in [0,1]$ and $u, v \in \mathbb{R}^n$ the inequality*

$$|F'(\tau u + (1-\tau)v)| \leq \mu(\tau)(G(u) + G(v)) \tag{A.5}$$

for some continuous and non-negative function G and some non-negative function $\mu \in L^1[0,1]$.
Under these assumptions the following estimate is true:

$$\|F(u)\|_{\dot{H}^s_r(\mathbb{R}^n)} \lesssim \|G(u)\|_{L^{r_1}(\mathbb{R}^n)} \|u\|_{\dot{H}^s_{r_2}(\mathbb{R}^n)} \tag{A.6}$$

for any $u \in \dot{H}^s_{r_2}(\mathbb{R}^n)$ such that $G(u) \in L^{r_1}(\mathbb{R}^n)$, provided that

$$\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}.$$

The reader can find the proof of this proposition in [2] and [53]. In particular we can apply Proposition A.3 for $F(u) = |u|^p$ or $F(u) = \pm|u|^{p-1}u$. For $G(u) = |F'(u)|$ and μ is a positive constant we obtain the following result.

Corollary A.2. *Let $F(u) = |u|^p$ or $F(u) = \pm|u|^{p-1}u$ for $p > 1, s \in (0,1)$ and $r, r_1, r_2 \in (1, \infty)$. Then,*

$$\|F(u)\|_{\dot{H}^s_r(\mathbb{R}^n)} \lesssim \|u\|^{p-1}_{L^{r_1}(\mathbb{R}^n)} \|u\|_{\dot{H}^s_{r_2}(\mathbb{R}^n)} \tag{A.7}$$

for any $u \in L^{r_1}(\mathbb{R}^n) \cap \dot{H}^s_{r_2}(\mathbb{R}^n)$, provided that

$$\frac{1}{r} = \frac{p-1}{r_1} + \frac{1}{r_2}.$$

The last corollary can the reader find it in [59] and [2] which is true only for $s \in (0,1)$. But we need to estimate the term $\|F(u)\|_{\dot{H}^s_r(\mathbb{R}^n)}$ for larger s. For this reason we recall the following result introduced and proved in [54].

Proposition A.4. *Let us choose $s > 0, p > \lceil s \rceil$ and $1 < r, r_1, r_2 < \infty$ satisfying*

$$\frac{1}{r} = \frac{p-1}{r_1} + \frac{1}{r_2}.$$

Let us denote by $F(u)$ one of the functions $|u|^p, \pm|u|^{p-1}u$.
Then it holds the following fractional chain rule:

$$\||D|^s F(u)\|_{L^r(\mathbb{R}^n)} \lesssim \|u\|^{p-1}_{L^{r_1}(\mathbb{R}^n)} \||D|^s u\|_{L^{r_2}(\mathbb{R}^n)}. \tag{A.8}$$

A.4. Fractional powers

Sobolev embedding is very useful in estimates for $\||u|^p\|_{\dot{H}^s_r(\mathbb{R}^n)}$, where $s > \frac{n}{r}$. We introduce from [57] and [13] the following results.

Proposition A.5. *Let $p > 1$ and $u \in H^s_m(\mathbb{R}^n)$, where $s \in \left(\frac{n}{m}, p\right)$. Then the following estimates hold:*

$$\||u|^p\|_{H^s_m(\mathbb{R}^n)} \lesssim \|u\|_{H^s_m(\mathbb{R}^n)} \|u\|^{p-1}_{L^\infty(\mathbb{R}^n)}, \tag{A.9}$$

$$\|u|u|^{p-1}\|_{H^s_m(\mathbb{R}^n)} \lesssim \|u\|_{H^s_m(\mathbb{R}^n)} \|u\|^{p-1}_{L^\infty(\mathbb{R}^n)}. \tag{A.10}$$

For the proof see [57]. We can derive from Proposition A.5 the following corollary.

Corollary A.3. *Under the assumptions of Proposition A.5 it holds:*

$$\||u|^p\|_{\dot{H}^s_m(\mathbb{R}^n)} \lesssim \|u\|_{\dot{H}^s_m(\mathbb{R}^n)} \|u\|^{p-1}_{L^\infty(\mathbb{R}^n)}, \tag{A.11}$$

$$\|u|u|^{p-1}\|_{\dot{H}^s_m(\mathbb{R}^n)} \lesssim \|u\|_{\dot{H}^s_m(\mathbb{R}^n)} \|u\|^{p-1}_{L^\infty(\mathbb{R}^n)}. \tag{A.12}$$

For the proof see [13] or [57].

Proposition A.6. *Let $0 < 2s^* < n < 2s$. Then for any function $f \in \dot{H}^{s^*}(\mathbb{R}^n) \cap \dot{H}^s(\mathbb{R}^n)$ one has*

$$\|f\|_{L^\infty(\mathbb{R}^n)} \leq \|f\|_{\dot{H}^{s^*}(\mathbb{R}^n)} + \|f\|_{\dot{H}^s(\mathbb{R}^n)}. \tag{A.13}$$

The following proof of this statement was given in [7].

Proof. Let us recall the following Sobolev's embeddings,

$$\|f\|_{L^\infty(\mathbb{R}^n)} \lesssim \|f\|_{H^\alpha_q(\mathbb{R}^n)}, \qquad \text{for} \qquad \alpha q > n,$$

and

$$\|f\|_{L^{\frac{2n}{n-2s^*}}(\mathbb{R}^n)} \lesssim \||D|^{s^*} f\|_{L^2(\mathbb{R}^n)}, \qquad \text{for} \qquad 0 < s^* < \frac{n}{2}.$$

If we fix $q = \frac{2n}{n-2s^*}$ and $\alpha = s - s^*$, then we get

$$\begin{aligned} \|f\|_{L^\infty(\mathbb{R}^n)} &\lesssim \|f\|_{L^{\frac{2n}{n-2s^*}}(\mathbb{R}^n)} + \||D|^\alpha f\|_{L^{\frac{2n}{n-2s^*}}(\mathbb{R}^n)} \\ &\lesssim \||D|^{s^*} f\|_{L^2(\mathbb{R}^n)} + \||D|^{\alpha+s^*} f\|_{L^2(\mathbb{R}^n)}. \end{aligned}$$

The proof is completed. □

A.5. Interpolation theory

Proposition A.7. *If $u \in \dot{H}^{s_1}(\mathbb{R}^n) \cap \dot{H}^{s_2}(\mathbb{R}^n)$, then we have for $s = (1-\theta)s_1 + \theta s_2, s \in [s_1, s_2]$ and $\theta \in (0,1)$ the following inequality:*

$$\|u\|_{\dot{H}^s(\mathbb{R}^n)} \lesssim \|u\|^{1-\theta}_{\dot{H}^{s_1}(\mathbb{R}^n)} \|u\|^{\theta}_{\dot{H}^{s_2}(\mathbb{R}^n)}.$$

For the proof one can see [1].

A.6. Some fixed point arguments

Proposition A.8. *The operator N maps $X(t)$ into itself and has one and only one fixed point $u \in X(t)$ if the following inequalities hold:*

$$\begin{aligned} \|Nu\|_{X(t)} &\leq C_0(t)\|(u_0,u_1)\|_{\mathcal{A}_{m,s}} + C_1(t)\|u\|^p_{X(t)}, &\text{(A.14)} \\ \|Nu - Nv\|_{X(t)} &\leq C_2(t)\|u-v\|_{X(t)}\big(\|u\|^{p-1}_{X(t)} + \|v\|^{p-1}_{X(t)}\big), &\text{(A.15)} \end{aligned}$$

where $C_1(t), C_2(t) \longrightarrow 0$ for $t \longrightarrow +0$ and $C_0(t), C_1(t), C_2(t) \leq C$ for all $t \in [0,\infty)$.

Proof. Let $L(t) := \|u\|_{X(t)}$. We prove that for any $\epsilon \in [0, \epsilon_0]$ we have

$$L(t) \leq C_0\epsilon + C_1(L(t))^p \ \text{ implies } \ L(t) \leq 2C_0\epsilon.$$

Let $\phi(x) := x - C_1 x^p$. Then $\phi(x) \leq x$ for any $x \geq 0$ and $\phi'(x) \geq \frac{1}{2}$ for $x \in \big[0, \overline{x}\big]$, where $\overline{x} = \big(\frac{1}{2C_1 p}\big)^{\frac{1}{p-1}}$. Consequently,

$$\phi(x) \leq x \leq 2\phi(x) \quad \text{for} \quad x \in [0, \overline{x}].$$

Let $\epsilon_0 = \min\{\overline{x}, \frac{\overline{x}}{2C_0}\}$. Then $L(0) \leq \overline{x}$ for $\|(u_0,u_1)\|_{\mathcal{A}_{1,1}} = \epsilon$. Thanks to (A.14) we get

$$\phi(L(t)) \leq C_0\epsilon \leq C_0\epsilon_0 \leq \frac{\overline{x}}{2} \leq \phi(\overline{x}).$$

So, $L(t) \in [0, \overline{x}]$ and $L(t) \leq 2\phi(L(t)) \leq 2C_0\epsilon$. This proves that N maps $X(t)$ into itself. Let us define the sequence $\{u_j\}_{j\geq 0}$ inductively by

$$u_0 \equiv 0 \quad \text{and} \quad u_j \equiv N(u_{j-1}).$$

Then $\|u_j\|_{X(t)} \leq 2C_0\epsilon = C\epsilon$, where $\epsilon \in [0, \epsilon_0]$ and C is independent of t. Applying (A.15) we have for $\epsilon_0 \leq \frac{1}{2C}$ sufficiently small the following estimate:

$$\begin{aligned} \big\|u_{j+1} - u_j\big\|_{X(t)} &= \|N(u_j) - N(u_{j-1})\|_{X(t)} \\ &\leq \tfrac{1}{2}\|u_j - u_{j-1}\|_{X(t)} \leq \cdots \leq \tfrac{1}{2^j}\|u_1\|_{X(t)}. \end{aligned}$$

This implies that $\{u_j\}_j$ is a Cauchy sequence in the Banach space $X(t)$. This sequence converges to the unique fixed point $N(u) = u$ for $t \in [0, \infty)$ since all the constants are independent of t. $\square$

Analogously, we can prove for $p, q > 1$ the following proposition which we need for the treatment of weakly coupled systems of semilinear damped waves.

Proposition A.9. *Let us suppose that for any $(u_0, u_1), (v_0, v_1) \in \mathcal{A}_{m_1,s_1} \times \mathcal{A}_{m_2,s_2}$ the mapping N satisfies the following estimates:*

$$\begin{aligned} \|N(u,v)\|_{X(t)} \leq C_0(t) \big(&\|(u_0,u_1)\|_{\mathcal{A}_{m_1,s_1}} + \|(v_0,v_1)\|_{\mathcal{A}_{m_2,s_2}}\big) \\ &+ C_1(t)\left(\|(u,v)\|^p_{X(t)} + \|(u,v)\|^q_{X(t)}\right), \end{aligned} \tag{A.16}$$

$$\begin{aligned} \|N(u,v) - N(\tilde{u},\tilde{v})\|_{X(t)} &\leq C_2(t)\|(u,v) - (\tilde{u},\tilde{v})\|_{X(t)} \\ &\times\big(\|(u,v)\|^{p-1}_{X(t)} + \|(\tilde{u},\tilde{v})\|^{p-1}_{X(t)} + \|(u,v)\|^{q-1}_{X(t)} + \|(\tilde{u},\tilde{v})\|^{q-1}_{X(t)}\big), \end{aligned} \tag{A.17}$$

where $C_1(t), C_2(t) \longrightarrow 0$ for $t \longrightarrow +0$ and $C_0(t), C_1(t), C_2(t) \leq C$ for all $t \in [0, \infty)$. Then N maps $X(t)$ into itself and has one and only one fixed point $(u,v) \in X(t)$.

Proof. Let us define $L(t) := \|(u,v)\|_{X(t)}$. At first we shall prove the following inequality:

$$\text{if } \; L(t) \leq C_0\epsilon + C_1(L(t))^{\max\{p;q\}} \quad \text{for any} \quad \epsilon \in [0, \epsilon_0], \;\text{ then }\; L(t) \leq 2C_0\epsilon.$$

Let $\phi(x) := x - C_1x^{\max\{p;q\}}$. Then $\phi(x) \leq x$ for any $x \geq 0$, and $\phi'(x) \geq \frac{1}{2}$ for $x \in \big[0, \overline{x}\big]$, where $\overline{x} = \big(\frac{1}{2C_1\max\{p;q\}}\big)^{\frac{1}{\max\{p;q\}-1}}$.
Consequently,

$$\phi(x) \leq x \leq 2\phi(x) \quad \text{for} \quad x \in [0, \overline{x}].$$

Let $\epsilon_0 = \min\{\overline{x}, \frac{\overline{x}}{2C_0}\}$. Then $L(0) \leq \overline{x}$ for $\|(u_0,u_1)\|_{\mathcal{A}_{m_1,s_1}} + \|(v_0,v_1)\|_{\mathcal{A}_{m_2,s_2}} = \epsilon$. Thanks to (A.16) we get

$$\phi(L(t)) \leq C_0\epsilon \leq C_0\epsilon_0 \leq \frac{\overline{x}}{2} \leq \phi(\overline{x}).$$

So, $L(t) \in [0, \overline{x}]$ and $L(t) \leq 2\phi(L(t)) \leq 2C_0\epsilon$. This proves that N maps $X(t)$ into itself. Let us define the sequence $\{(u_j, v_j)\}_{j\geq 0}$ inductively by

$$(u_0, v_0) \equiv (0,0) \quad \text{and} \quad (u_j, v_j) \equiv N(u_{j-1}, v_{j-1}).$$

Then $\|u_j, v_j)\|_{X(t)} \leq 2C_0\epsilon = C\epsilon$, where $\epsilon \in [0, \epsilon_0]$ and C is independent of t. Using (A.17) for $\epsilon_0 \leq \frac{1}{2C}$ sufficiently small we obtain the following estimate:

$$\begin{aligned}\|(u_{j+1}, v_{j+1}) - (u_j, v_j)\|_{X(t)} &= \|N(u_j, v_j) - N(u_{j-1}, v_{j-1})\|_{X(t)} \\ &\leq \tfrac{1}{2}\|(u_j, v_j) - (u_{j-1}, v_{j-1})\|_{X(t)} \leq \cdots \leq \tfrac{1}{2^j}\|(u_1, v_1)\|_{X(t)}.\end{aligned}$$

This implies that $\{(u_j, v_j)\}_j$ is a Cauchy sequence in the Banach space $X(t)$ converging to the unique fixed point $N(u, v) = (u, v)$ for $t \in [0, \infty)$ since all the constants are independent of t. $\square$

List of symbols and abbreviations

Symbols

$\lvert x\rvert$	Euclidean norm of $x \in \mathbb{R}^n$,
$\lceil x\rceil$	Ceiling, the smallest integer greater than or equal to x,
$\lfloor x\rfloor$	Floor, the largest integer less than or equal to x,
$[x]^+$	$\max\{x; 0\}$,
$\nabla = \nabla_x = (\partial_{x_1}, \cdots, \partial_{x_1})$	Gradient,
$\Delta = \Delta_x = \sum_{j=1}^n \partial_{x_j}^2$	Laplacian with respect to $x \in \mathbb{R}^n$,
$\lvert D\rvert^s = \lvert D_x\rvert^s$	Pseudo-differential operator with symbol $\lvert\xi\rvert^s$,
$f \lesssim g$	If $f \leq cg$ for some constant $c \in \mathbb{R}^+$,
$f \gtrsim g$	If $f \geq cg$ for some constant $c \in \mathbb{R}^+$,
$f \approx g$	If $f \lesssim g$ and $f \gtrsim g$,
$\lVert L\rVert_{A\longrightarrow B}$	the norm of a linear mapping defined in A with values in B,
$p_{Fuj}(n)$	$p_{Fuj}(n) := 1 + \frac{2}{n}$ Fujita exponent,
$p_{Fuj,m}(n)$	$p_{Fuj,m}(n) := 1 + \frac{2m}{n}$ modified Fujita exponent,
$p_{GN}(n)$	$p_{GN}(n) := \frac{n}{n-2}$ Gagliardo–Nirenberg exponent,
$p_{GN,s}(n)$	$p_{GN,s}(n) := \frac{n}{n-2s}$ modified Gagliardo–Nirenberg exponent.

Function spaces

$\mathcal{C}^k(\mathbb{R}^n)$	space of $k-$times continuously differentiable functions,
$\mathcal{C}^\infty(\mathbb{R}^n)$	space of infinitely continuously differentiable functions,
$\mathcal{C}_0^\infty(\mathbb{R}^n)$	space of functions belonging to $\mathcal{C}^\infty(\mathbb{R}^n)$ with compact support,
$L^p(\mathbb{R}^n)$	$L^p(\mathbb{R}^n) = \left\{u : \mathbb{R}^n \longrightarrow \mathbb{R} : \int_{\mathbb{R}^n} \lvert u(x)\rvert^p dx < \infty\right\}$, where the functions u are Lebesgue measurable and $1 \leq p < \infty$,
$\lVert\cdot\rVert_{L^p(\mathbb{R}^n)}$	denote L^p-norms where $\lVert u\rVert_{L^p(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} \lvert u(x)\rvert^p dx\right)^{\frac{1}{p}}$,
$L^\infty(\mathbb{R}^n)$	$L^\infty(\mathbb{R}^n) = \{u : \mathbb{R}^n \longrightarrow \mathbb{R} : \text{ess sup}_{x\in\mathbb{R}^n}\lvert u(x)\rvert < \infty\}$,
$\lVert\cdot\rVert_{L^\infty(\mathbb{R}^n)}$	denote $L^\infty-$norms where, $\lVert u\rVert_{L^\infty(\mathbb{R}^n)} = \text{ess sup}_{x\in\mathbb{R}^n}\lvert u(x)\rvert$,
$H_p^m(\mathbb{R}^n)$	$H_p^m(\mathbb{R}^n) = \{u \in L^p(\mathbb{R}^n) : \partial_x^\alpha u \in L^p(\mathbb{R}^n) \quad \text{for all} \quad \alpha \in \mathbb{N}^n \quad \text{with} \quad \lvert\alpha\rvert \leq m\}$ where the functions u are Lebesgue measurable, $m \in \mathbb{N}$ and $1 \leq p \leq \infty$,
$\lVert\cdot\rVert_{H_p^m(\mathbb{R}^n)}$	$\lVert u\rVert_{H_p^m(\mathbb{R}^n)} = \sum_{\lvert\alpha\rvert\leq m} \left(\int_{\mathbb{R}^n} \lvert\partial_x^\alpha u(x)\rvert^p\right)^{\frac{1}{p}} \quad \text{for all} \quad 1 \leq p < \infty$,
$\lVert\cdot\rVert_{H_\infty^m(\mathbb{R}^n)}$	$\lVert u\rVert_{H_\infty^m(\mathbb{R}^n)} = \max_{\lvert\alpha\rvert\leq m} \text{ess sup}_{x\in\mathbb{R}^n} \lvert\partial_x^\alpha u(x)\rvert$,
$H^m(\mathbb{R}^n)$	denote $H_p^m(\mathbb{R}^n)$, where $p = 2$ i.e. $H^m(\mathbb{R}^n) := H_2^m(\mathbb{R}^n)$,
$\dot{H}_p^s(\mathbb{R}^n)$	homogeneous Sobolev space based on $L^p(\mathbb{R}^n), s \in \mathbb{R}$,
$\dot{H}^s(\mathbb{R}^n) = \dot{H}_2^s(\mathbb{R}^n)$	homogeneous Sobolev space based on $L^2(\mathbb{R}^n), s \in \mathbb{R}$.

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